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ON AN INEQUALITY OF P. ERDÖS, J. NEVEU, A. RÉNYI AND S. ZUBRZYCKI

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1. Introduction. Erdös et al. [1] proved the following:

For any sequence $\{A_i, 1 \leqslant i \leqslant n\}$ *of* n *events in an arbitrary probability space* $(S, \mathscr{A}, P)$ *such that* $P(A_i A_j) \leqslant a^2$ *for all* i, j $(i \neq j)$ *the inequality*

$$\sum_{i=1}^{n} \big(P(A_i) - a\big) \leqslant \varepsilon \tag{1}$$

holds, where

$$\varepsilon = \frac{1-a}{2} + \frac{(na-r)\big((n-1)a-r\big)}{2r},$$

and r *is the largest integer satisfying*

$$r(r-1) \leqslant [n(n-1)a^2],$$

$[x]$ *denoting the integral part of* x.

They also establish that ε is of order $1/n$.

For a similar problem, Zubrzycki [3] shows that if $A_1, A_2, \ldots, A_n$ are exchangeable events of second order, i.e., $P(A_i) = v_1$ $(i = 1, 2, \ldots, n)$, $P(A_i A_j) = v_2$ $(i \neq j)$, then

$$v_2 \geqslant v_1^2 - \frac{v_1(1-v_1)}{n-1} + \frac{(nv_1 - [nv_1])(1 - nv_1 + [nv_1])}{n(n-1)}. \tag{2}$$

In the above-mentioned papers the authors give examples where equality is attained in (1) and (2). The results of [1] are equivalent to those obtained from exchangeable events of second order, although the authors give the impression that this is not the case.

In the present paper we derive inequalities (1) and (2) using linear programming approach and, furthermore, we study the more general case of non-exchangeable events. We improve inequality (1) and we present examples where our inequalities are sharper than (1). Our approach is a consequence of a result of Rényi [2], who proved the following theorem:

Let $F_j = F_j(A_1, A_2, \ldots, A_n)$ $(j = 1, 2, \ldots, N)$ *be arbitrary Boolean functions of the n variable events* $A_1, A_2, \ldots, A_n$. *The linear inequality*

$$\sum_{j=1}^{N} c_j P(F_j) \geqslant 0$$

(*where* $c_1, c_2, \ldots, c_N$ *are real constants*) *is valid in every probability space if it is valid in the trivial probability space* $S_1(1)$.

The trivial probability space consists only of events having probability zero or one.

In this paper we also prove (Theorem 1), using indicator random variables, a result similar to the theorem of Rényi, which is then utilized to generalize the results of [1] and [2].

2. Main results.

THEOREM 1. *Let* $F_j = F_j(A_1, A_2, \ldots, A_n)$ $(j = 1, 2, \ldots, N)$ *be arbitrary Boolean functions of the variable events* $A_1, A_2, \ldots, A_n$. *Then*

(i) $\sum_{j=1}^{N} c_j I_j(w) \geqslant 0$ *for every* $w \in S$

if and only if

(ii) $\sum_{j=1}^{N} c_j P(F_j) \geqslant 0$ *in any probability space* $(S, \mathscr{A}, P)$,

where c_j *are real constants and* $I_j(w) = I_j$ *denotes the indicator random variable of* F_j $(j = 1, 2, \ldots, N)$.

Proof. We can assume that $F_1 \neq \emptyset, F_2 \neq \emptyset, \ldots, F_N \neq \emptyset$. Let $E_1, E_2, \ldots, E_r$ be all the non-null atoms of $A_1, A_2, \ldots, A_n$, $r \leqslant 2^n$, and

$$d_i = \sum_{\{j: E_i \subset F_j\}} c_j.$$

Then our theorem is equivalent to:

(i′) $\sum_{j=1}^{r} d_j I_{E_j}(w) \geqslant 0$ for all $w \in S$

if and only if

(ii′) $\sum_{j=1}^{r} d_j P(E_j) \geqslant 0$ in every probability space.

The necessity is proved by taking expectations.

To prove the sufficiency consider the probability law

$$P(E_j) = \begin{cases} 1 & \text{for } j = j_0, \\ 0 & \text{for } j \neq j_0, \end{cases}$$

where $1 \leqslant j \leqslant r$.

Then $d_{j_0}\cdot 1 \geqslant 0$ or $d_{j_0} \geqslant 0$, and this happens for all $j_0 = 1, 2, \dots, r$. Hence we have (i′).

It should be noted that if (i) holds, then so does (ii), however, to go from (ii) to (i) we should make the basic assumption that (ii) holds in every probability space. It is evident from the proof above that

$$\sum_{j=1}^{N} c_j I_j(w) = 0 \quad \text{for all } w \in S \tag{3}$$

if and only if

$$\sum_{j=1}^{N} c_j P(F_j) = 0 \quad \text{in every probability space.} \tag{4}$$

The results above will be used in order to establish some inequalities generalizing the results of [1] and [2]. Let us use the notation: $P_i = P(A_i)$ and $P_{ij} = P(A_i A_j)$ $(1 \leqslant i < j \leqslant n)$.

THEOREM 2. *For any events $A_1, A_2, \dots, A_n$, the inequalities*

$$\sum_{i=1}^{n} P_i \leqslant \frac{1}{r^*} \sum_{i<j} P_{ij} + \frac{r^*+1}{2}, \tag{5}$$

$$k^* \sum_{i=1}^{n} P_i - \frac{k^*(k^*+1)}{2} \leqslant \sum_{i<j} P_{ij} \tag{6}$$

are valid in any probability space, where r^* is the largest integer satisfying

$$r^*(r^*-1) \leqslant \Big[2 \sum_{i<j} P_{ij}\Big], \quad \textit{and } k^* = \Big[\sum_{i=1}^{n} P_i\Big].$$

Proof. Assume that the inequality

$$\sum_{i=1}^{n} P_i \leqslant \sum_{i<j} c_{ij} P_{ij} + c \tag{7}$$

is valid in any probability space. Then by Theorem 1 we have

$$\sum_{i=1}^{n} I_i(w) \leqslant \sum_{i<j} c_{ij} I_i(w) I_j(w) + c \quad \text{for all } w \in S, \tag{8}$$

where $I_i(w)$ is the indicator of the event A_i $(i = 1, 2, \dots, n)$.

Relation (8) is certainly true if

$$r \leqslant \sum_{\substack{i,j \in J_r \\ i<j}} c_{ij} + c \quad \text{for all } r = 0, 1, \dots, n, \tag{9}$$

where J_r is a subset of $\{1, 2, \dots, n\}$ containing r integers.

To see this, let us take $w \in S$. Then w belongs to exactly r of the events $A_1, A_2, \ldots, A_n$ for some $r = 0, 1, \ldots, n$, and hence (8) holds for every $w \in S$. Of course, for a particular collection of events $A_1, A_2, \ldots, A_n$ some of relations (9) might not be needed; in this case we have fewer inequality constraints and the bounds given in (5) and (6) could be improved as in Corollary 1.

Our problem is to minimize the right-hand side of (7) subject to (9). Consider the case where $c_{ij} = c_1$ for all $1 \leqslant i < j \leqslant n$. Then we have to minimize

$$c_1 \sum_{i<j} P_{ij} + c \tag{10}$$

subject to

$$r \leqslant \frac{r(r-1)}{2} c_1 + c \quad (r = 1, 2, \ldots, n). \tag{11}$$

Relations (11) define a convex polygon with vertices

$$c_1 = \frac{1}{r}, \quad c = \frac{r+1}{2} \quad (r = 1, 2, \ldots, n-1),$$

as it is easily seen from the (c_1, c)-plane. The value of (10) at these vertices is

$$Z_r = \frac{1}{r} \sum_{i<j} P_{ij} + \frac{r+1}{2} \quad (r = 1, 2, \ldots, n-1).$$

Observe that $Z_r \leqslant Z_{r+1}$ for $r \geqslant r^*$ and $Z_{r-1} \geqslant Z_r$ for $r \leqslant r^*$, whence we obtain (5).

To obtain (6) we argue similarly maximizing

$$d_1 \sum_{i=1}^{n} P_i + d$$

subject to

$$\frac{r(r-1)}{2} \geqslant r d_1 + d \quad (r = 0, 1, \ldots, n).$$

The vertices here are $d_1 = k$, $d = -k(k+1)/2$, with $k^* = [\sum_{i=1}^{n} P_i]$ for the optimal vertex.

COROLLARY 1. *If every $w \in S$ belongs either to r_1 or to r_2 of the events $A_1, A_2, \ldots, A_n$, then*

$$\sum_{i=1}^{n} P_i = \frac{1}{r_1 + r_2 - 1} \Big(2 \sum_{i<j} P_{ij} + r_1 r_2 \Big). \tag{12}$$

Proof. For the given events the relations

$$r_1 = \binom{r_1}{2} c_1 + c, \quad r_2 = \binom{r_2}{2} c_1 + c \tag{13}$$

and

$$\sum_{i=1}^{n} I_i(w) = c_1 \sum_{i<j} I_i(w) I_j(w) + c \quad \text{for every } w \in S \tag{14}$$

are equivalent. From (13) we get

$$c_1 = \frac{2}{r_1+r_2-1}, \quad c = \frac{r_1 r_2}{r_1+r_2-1},$$

and by taking expectations in (14) we obtain (12).

An obvious lower bound for $\sum_{i=1}^{n} P_i$ is given by

$$(n-1) \sum_{i=1}^{n} P_i \geqslant 2 \sum_{i<j} P_{ij}. \tag{15}$$

COROLLARY 2. *In any probability space we have*

$$\sum_{i=1}^{n} P_i + P\Big(\bigcap_{i=1}^{n} \bar{A}_i\Big) \leqslant \frac{1}{r^*} \sum_{i<j} P_{ij} + \frac{r^*+1}{2},$$

where $\bar{A}$ denotes the complement of A.

For the proof we have to minimize the same linear function given in (7) subject to the same restrictions (9).

In the case of exchangeable events of second order, (5), (6) and (15) take the forms

$$v_1 \leqslant \frac{n-1}{2r^*} v_2 + \frac{r^*+1}{2n}, \quad v_2 \geqslant \frac{2k^* v_1}{n-1} - \frac{k^*(k^*+1)}{n(n-1)}, \quad v_1 \geqslant v_2, \tag{16}$$

where r^* is the largest integer satisfying

$$r^*(r^*-1) \leqslant [n(n-1)v_2],$$

and $k^* = [nv_1]$. For large n we have

$$v_1 \leqslant \frac{n-1}{2r^*} v_2 + \frac{r^*+1}{2n} = \sqrt{v_2} + O\left(\frac{1}{n}\right) \to \sqrt{v_2} \quad \text{as } n \to \infty,$$

$$v_2 \geqslant \frac{2k^* v_1}{n-1} - \frac{k^*(k^*+1)}{n(n-1)} = v_1^2 + O\left(\frac{1}{n}\right) \to v_1^2 \quad \text{as } n \to \infty,$$

and $v_1 \geqslant v_2$ for all n.

We now construct n events $A_1, A_2, \ldots, A_n$ such that every $w \in S$ belongs either to r_1 or to r_2 of them ($r_1 \neq r_2$). For this

(i) take any m events B_i, where

$$m = \binom{n}{r_1} + \binom{n}{r_2}, \quad i = 1, 2, \ldots, m, \quad B_i \cap B_j = \emptyset, \ \bigcup_{i=1}^{m} B_i = S;$$

(ii) arrange in the lexicographical order all combinations of r_1 and all combinations of r_2 of different integers among $1, 2, \ldots, n$;

(iii) define the event A_k ($k = 1, 2, \ldots, n$) so that it contains B_i if the i-th combination (word) contains the integer k.

From this construction we see that:

(a) every B_i belongs to all the events A_k for which the index k is contained in the i-th combination, i.e., every B_i is contained either in r_1 or in r_2 of the A_k ($k = 1, 2, \ldots, n$);

(b) every A_k contains as many events B_i as the number of different combinations (words) containing the index k, i.e.,

$$\binom{n-1}{r_1-1} + \binom{n-1}{r_2-1};$$

(c) every pair of events A_k, A_p has in common as many events B_i as the number of combinations (words) containing the indices k and p, i.e.

$$\binom{n-2}{r_1-2} + \binom{n-2}{r_2-2}.$$

If we now assign to each B_i the probability q_i such that

$$\sum_{i=1}^{m} q_i = 1,$$

then equality in (12) is attained, since

$$\sum_{i=1}^{n} P(A_i) = r_1 Q_1 + r_2 Q_2, \qquad \sum_{i<j} P(A_i A_j) = \binom{r_1}{2} Q_1 + \binom{r_2}{2} Q_2,$$

where

$$Q_1 = P(B_1) + \ldots + P(B_{j^*}), \quad j^* = \binom{n}{r_1}, \quad \text{and} \quad Q_2 = 1 - Q_1.$$

The following example will clarify the situation.

Example 1. Take $n = 3, r_1 = 1, r_2 = 2$. Then all possible combinations (words) of $\{1, 2, 3\}$ of length 1 and 2 are: $(1), (1,2), (1,3), (2), (2,3), (3)$.

Thus

$$A_1 = B_1 \cup B_2 \cup B_3, \quad A_2 = B_2 \cup B_4 \cup B_5, \quad A_3 = B_3 \cup B_5 \cup B_6.$$

If we set

$$P(B_1) = 0.35, \quad P(B_2) = 0.10, \quad P(B_3) = 0.05, \quad P(B_4) = 0.15,$$
$$P(B_5) = 0.20, \quad P(B_6) = 0.15,$$

we get

$$P(A_1) = 0.50, \quad P(A_2) = 0.45, \quad P(A_3) = 0.40,$$
$$P(A_1A_2) = 0.10, \quad P(A_1A_3) = 0.05, \quad P(A_2A_3) = 0.20$$

and

$$\sum_{i=1}^{3} P(A_i) = 1.35, \quad \sum_{i<j} P(A_iA_j) = 0.35, \quad r_1 = 1, r_2 = 2.$$

By (5) or (12) we have

$$\frac{1}{1+2-1}(2 \cdot 0.35 + 2) = 1.35.$$

Also equality in (6) is attained.

In the proof of Theorem 2, we considered only the special case where $c_{ij} = c_1$ $(1 \leqslant i < j \leqslant n)$ and the derived bounds are certainly inferior to those resulting from the unrestricted c_{ij}. Although we cannot give the best bounds in analytical form, nevertheless in the non-exchangeable case we can improve the bound given by (5).

For exchangeable events of second order, inequalities (5), (6) and (15) cannot be improved using the method of Theorem 1, but if we drop the assumption of exchangeability, then some progress can be made.

To do this we split the set $\{1, 2, \ldots, n\}$ into two disjoint sets J_s and J'_t having s and $t = n-s$ integers, respectively, and put $c_{ij} = c_1$ if $i, j \in J_s$, $c_{ij} = c_2$ if $i, j \in J'_t$, $c_{ij} = c_3$ if $i \in J_s, j \in J'_t$. Then (9) takes the form

$$r_1 + r_2 \leqslant \frac{r_1(r_1-1)}{2} c_1 + \frac{r_2(r_2-1)}{2} c_2 + r_1 r_2 c_3 + c, \tag{17}$$

$$0 < r_1 \leqslant s, \ 0 \leqslant r_2 \leqslant n-s.$$

The point $c_1 = 0, c_2 = 0, c_3 = 1/c, c = \max(s, n-s)$ is an interior point or a vertex of the convex polyhedron defined by (17).

Now from (7) we obtain

$$\sum_{i=1}^{n} P_i \leqslant \frac{1}{c} Q_s + c,$$

where $Q_s = \sum P_{ij}$ with $i \in J_s$, $j \in J'_t$, $c = \max(s, n-s)$. Hence, taking all possible $s = 1, 2, \ldots, n-1$, we finally have

$$\sum_{i=1}^{n} P_i \leqslant \min_s \left(\frac{1}{c} Q_s + c\right). \tag{18}$$

In the sequel we give an algorithm of how to determine, for given s, the choice of J_s so that Q_s is minimized, but first we present an example in which (18) gives a better bound than (5) and although the example is trivial it demonstrates our purpose.

Example 2. Take in the sample space S the events D_1, D_2, D_3 to be mutually disjoint and

$$D_1 \cup D_2 \cup D_3 = S \quad \text{with} \quad P(D_1) = 0.30, \quad P(D_2) = 0.25, \quad P(D_3) = 0.45.$$

Define now the events A_1, A_2, A_3, A_4 as follows:

$$A_1 = A_2 = D_1 \cup D_2 \quad \text{and} \quad A_3 = A_4 = D_1 \cup D_3.$$

Then

$$P_1 = P_2 = 0.55, \quad P_3 = P_4 = 0.75, \quad P_{12} = 0.55,$$

$$P_{13} = P_{14} = P_{23} = P_{24} = 0.30, \quad P_{34} = 0.75.$$

Thus

$$\sum_{i=1}^{4} P_i = 2.60, \quad \sum_{i<j} P_{ij} = 2.50.$$

The upper bound of (5) is $2.50/2 + 3/2 = 2.75$ (since $r^* = 2$) and that of (18) is $1.20/2 + 2 = 2.60$ for the choice $J_2 = \{1, 2\}$, and this last bound equals the actual value of $\sum P_i = 2.60$.

We now describe the algorithm for finding the subset J_s, for given s, so that Q_s is minimized:

(i) For fixed s and $t = n - s$ take any two disjoint subsets J_s and J'_t such that $J_s \cup J'_t = \{1, 2, \ldots, n\}$.

(ii) Find the point $i^* \in J_s$ for which

$$R_1 = \sum_{\substack{j \in J_s \\ j \neq i^*}} P_{i^*j} - \sum_{j \in J'_t} P_{i^*j}$$

is maximized (ties are broken arbitrarily).

(iii) Find the point $j^* \in J'_t$ for which

$$R_2 = \sum_{i \in J_s} P_{ij^*} - \sum_{\substack{i \in J'_t \\ i \neq j^*}} P_{ij^*}$$

is minimized (ties are broken arbitrarily).

(iv) If $R_1 < R_2$, then the point i^* goes to J'_t and j^* goes to J_s.

(v) If $R_1 \geqslant R_2$, then for the subdivision J_s, J'_t the value of Q_s is minimized.

Having found the minimal Q_s for $s = 1, 2, \ldots, n-1$ we now proceed to find

$$\min_s \left(\frac{1}{c} Q_s + c\right) \quad \text{where } c = \max(s, n-s).$$

To verify that the algorithm above is valid, we take the two subsets J_s and J'_t and determine how the value of Q_s changes by transferring the point i^* into J'_t and the point j^* into J_s. This procedure leads us to the algorithm above.

In what follows we derive another upper bound for $\sum_{i=1}^{n} P_i$. For this we divide the set $\{1, 2, \ldots, n\}$ into three mutually exclusive subsets J_s, J_t and J_u having s, t and $u = n-s-t$ integers, respectively, with $s \leqslant [n/2]$, $t \leqslant [n/2]$ and $u \leqslant [n/2]$. If we set $c_{ij} = 0$ for $i, j \in J_s$ or for $i, j \in J_t$ or for $i, j \in J_u$, $c_{ij} = c_1$ for $i \in J_s$, $j \in J_t$, $c_{ij} = c_2$ for $i \in J_s$, $j \in J_u$, and $c_{ij} = c_3$ for $i \in J_t, j \in J_u$, then (9) takes the form

$$r_1 + r_2 + r_3 \leqslant c_1 r_1 r_2 + c_2 r_1 r_3 + c_3 r_2 r_3 + c \tag{19}$$
$$\text{for all } 0 \leqslant r_1 \leqslant s,\ 0 \leqslant r_2 \leqslant t,\ 0 \leqslant r_3 \leqslant u.$$

The values

$$c_1 = \frac{n-2u}{2st}, \quad c_2 = \frac{n-2t}{2su}, \quad c_3 = \frac{n-2s}{2tu} \quad \text{and} \quad c = \frac{n}{2}$$

satisfy (19), as it can be seen from the relation

$$c_1(r_1-s)(r_2-t) + c_2(r_1-s)(r_3-u) + c_3(r_2-t)(r_3-u) \geqslant 0.$$

Hence, another upper bound is

$$\sum_{i=1}^{n} P_i \leqslant \min_{s,t,u} \left(\frac{n-2u}{2st} Q_{s,t} + \frac{n-2t}{2su} Q_{s,u} + \frac{n-2s}{2tu} Q_{t,u}\right) + \frac{n}{2}, \tag{20}$$

where $Q_{s,t} = \sum P_{ij}$ for $i \in J_s$, $j \in J_t$, $Q_{s,u} = \sum P_{ij}$ for $i \in J_s$, $j \in J_u$, $Q_{t,u} = \sum P_{ij}$ for $i \in J_t, j \in J_u$, and $s+t+u = n$.

In practice it is a challenging computational problem to find the minimum of the right-hand side of inequality (20), however, we can develop an algorithm similar to the one developed before.

We present now an example where the bound given by (20) is better than that of (5) and (18).

Example 3. Take $n = 5$ and define the events A_1, A_2, A_3, A_4 as in Example 2. Further, put

$$A_5 = A_3 = A_4 = D_1 \cup D_3.$$

We have then

$$\sum_{i=1}^{5} P(A_i) = 3.35, \qquad \sum_{i<j} P(A_i A_j) = 4.60.$$

The bounds given by (5), (18) and (20) are 3.533, 3.60 and 3.475, respectively, where to evaluate (18) we took $J_2 = \{A_1, A_2\}, J_3 = \{A_3, A_4, A_5\}$ and to evaluate (20) we took $J_2 = \{A_1, A_2\}, J_2' = \{A_3, A_4\}, J_1 = \{A_5\}$.

COROLLARY 3. *If we divide the set* $J = 1, 2, \ldots, n$ *into* m *disjoint subsets* $J_1, J_2, \ldots, J_m$ *having* $n_1, n_2, \ldots, n_m$ *elements, respectively, and if* $Q_{s,t} = \sum P_{ij}$ *for* $i \in J_s$, $j \in J_t$ $(1 \leqslant s, t \leqslant m)$, *then*

$$\sum_{i=1}^{n} P_i \leqslant \min_{J_1, \ldots, J_m} \frac{1}{d(m-1)} \sum_{s<t} Q_{s,t} + \frac{dm}{2}, \tag{21}$$

where $d = \max(n_1, n_2, \ldots, n_m)$.

Proof. If we set $c_{ij} = 0$ for $i, j \in J_s$ $(s = 1, 2, \ldots, m)$, $c_{ij} = 1/d(m-1)$, $c = dm/2$, we see that (9) is satisfied for all $r = 1, 2, \ldots, n$, since

$$\frac{1}{d(m-1)} \sum_{i<j} (r_i - d)(r_j - d) \geqslant 0, \qquad 0 \leqslant r_i \leqslant n_i.$$

Finally, as a consequence of Corollary 2, we have

COROLLARY 4. *If* $J_1, J_2, \ldots, J_m$ *are disjoint subsets of* $J = \{1, 2, \ldots, n\}$ *such that* $J_1 \cup J_2 \cup \ldots \cup J_m = J$, *then*

$$\sum_{i=1}^{n} P_i + \sum_{s=1}^{m} P\Big(\bigcup_{i \in J_s} \bar{A}_i\Big) \leqslant \min_{J_1, \ldots, J_m} \left(\sum_{s=1}^{m} \frac{Q_{s,s}}{r_s^*} + \frac{1}{2} \sum_{s=1}^{m} r_s^* + \frac{m}{2} \right), \tag{22}$$

where $Q_{s,s} = \sum P_{i,j}$, $i, j \in J_s$, $i < j$, *and* r_s^* *is the largest integer satisfying* $r_s^*(r_s^* - 1) \leqslant 2Q_{s,s}$.

Proof. Since

$$\bigcup_{i \in J} A_i = \bigcup_{s=1}^{m} \Big(\bigcup_{i \in J_s} A_i \Big) \quad \text{and} \quad P\Big(\bigcup_{i \in J} A_i\Big) \leqslant \sum_{s=1}^{m} \Big(\bigcup_{i \in J_s} A_i \Big),$$

the result follows from Corollary 2.

The practical difficulty in evaluating the right-hand side of (21) or (22) is a demanding computational problem. However, we can start with $m = 2$, follow a procedure similar to the one already described for evaluating (18), and then proceed to $m = 3, 4, \ldots$ as long as the right-hand side of (21) or (22) decreases.

COROLLARY 5. *If we can divide the set* $J = \{1, 2, \ldots, n\}$ *into two disjoint subsets* J_1 *and* J_2 *with* s *and* $n - s$ *elements, respectively* $(s \leqslant [n/2])$ *and*

(a) *if* $P_{ij} \leqslant a_1^2$ *for all* $i \in J_1, j \in J_2$, *then*

$$\sum_{i=1}^{n} P_i \leqslant n - s(1-a_1^2); \tag{23}$$

(b) *if* $P_{ij} \leqslant a_1^2$ *for all* $i, j \in J_1$ *and* $i, j \in J_2$, *then*

$$\sum_{i=1}^{n} P_i \leqslant (n-1)a_1 + 1 + O\left(\frac{1}{s}\right) + O\left(\frac{1}{n-s}\right). \tag{24}$$

Proof. (a) It is enough to put $P_{ij} \leqslant a_1^2$ in (18).

(b) We have

$$Q_{11} = \frac{s(s-1)a_1^2}{2}, \quad Q_{22} = \frac{(n-s)(n-s-1)a_1^2}{2}$$

and, for large n_s,

$$r_s^* = n_s a_1 + \frac{1-a_1}{2} + O\left(\frac{1}{n_s}\right) \quad \text{and} \quad \frac{1}{r_s^*} = \frac{1}{n_s a_1}\left[1 - \frac{1-a_1}{2n_s a_1} + O\left(\frac{1}{n_s^2}\right)\right].$$

Hence

$$\frac{Q_{ss}}{r_s^*} + \frac{r_s^*+1}{2} = n_s a_1 + \frac{1-a_1}{2} + O\left(\frac{1}{n_s}\right),$$

where $n_1 = s$ and $n_2 = n-s$, which implies (24).

Of course, we can extend Corollary 5 along the lines of (21) and (22). It should be noticed that for the results of Corollary 5 we do not assume $P_{ij} \leqslant a^2$ for all $i \neq j$ but that $P_{ij} \leqslant a_1^2$ for some i, j. If $P_{ij} \leqslant a^2$ for all $i \neq j$, then we get

$$\sum_{i=1}^{n} P_i \leqslant na + \frac{1}{2}(1-a) + O\left(\frac{1}{n}\right)$$

given in [1], and this bound is worse than (23) or (24) if a_1 is small when compared to a.

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