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ON GRADIENT FIELDS

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Let $f: U \to R$, where $U \subset R^n$ is open, be a C^p-function, $p \geqslant 1$. Let $x \in U$ be a singular point of gradient field $\operatorname{grad} f$. We consider the following question proposed by R. Thom (oral communication of V. I. Arnold):

Does there exist a trajectory $\gamma(t)$ of $\operatorname{grad} f$ such that

$$(1) \qquad \lim_{t\to\infty}\gamma(t) = x, \ \lim_{t\to\infty}\frac{\gamma'(t)}{\|\gamma'(t)\|} = a \quad \text{or} \quad \lim_{t\to-\infty}\gamma(t) = x, \ \lim_{t\to-\infty}\frac{\gamma'(t)}{\|\gamma'(t)\|} = a,$$

where $\|\cdot\|$ is the Euclidean norm in R^n, $a \in R^n$?

THEOREM. *Let* $f: U \to R$, $U \subset R^n$ *open, be an analytic function and let* $x \in U$ *be a singular point of* $\operatorname{grad} f$. *Then there exists a trajectory of* $\operatorname{grad} f$ *with properties* (1).

Proof (based on the Curve Selection Lemma for semianalytic sets (1)). Suppose that $x = 0 \in R^n$, $f(0) = 0$. Consider a semianalytic set V defined by

$$(2) \qquad V = \{x \in U: \operatorname{grad} f(x) \neq 0,\ \operatorname{grad} f(x) \| x\},$$

where $\|$ denotes linearly dependent vectors. Then

$$(3) \qquad 0 \in \overline{V},$$

since the set of regular values R_f of the function f is dense in R.

For each $c \in R_f$ the set $M_c = \{x \in R^n: f(x) = c\}$ is a closed submanifold in R^n. Then the function $y = \|x\|^2$, $x \in M_c$, has a minimum for some $x_c \in M_c$. Hence we have $x_c \| \operatorname{grad} f(x_c)$. This proves (3).

Now, by the Curve Selection Lemma there exists an analytic curve with the properties

$$(4) \qquad \varphi: (-\varepsilon, \varepsilon) \to V, \quad \varphi(0) = 0, \quad \varphi\big((0, \varepsilon)\big) \subset V.$$

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(1) See F. Bruhat et H. Cartan, *Sur la structure des sous-ensembles analytiques réels*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris, Série A, 244 (1957), p. 988-990.

By the definition of V we have

(5) $$\operatorname{grad} f(\varphi(t)) \| \varphi(t), \quad \operatorname{grad} f(\varphi(t)) \neq 0, \quad \lim_{t\to 0+} \frac{\varphi(t)}{\|\varphi(t)\|} = a \neq 0.$$

For a small $r > 0$ the intersection $\varphi \cap S_r$ of the curve φ with the sphere $\{\|x\| = r\}$ is a single point p_r. Let $b \neq a$ be a vector linearly independent of $\varphi(t)$ for $0 < t < \eta$. Put $\psi(t) = tb$ for $t > 0$ and $q_r = \psi \cap S_r$. Consider a local diffeomorphism Φ defined as follows:

(6) $$\begin{aligned} &\Phi|_{S_r}: S_r \to S_r \text{ is the rotation in the plane } \{p_r, q_r\}, \\ &\Phi(q_r) = p_r, \quad \Phi(0) = 0. \end{aligned}$$

Construction of Φ. By (4) and (5) there exists a C^∞ regular parametrization $\varphi(s)$ of φ, $\varphi'(0) = a$. Consequently, for small s the equality $r(s) = \|\varphi(s)\|$, $s \geqslant 0$, defines a regular parametrization of φ in some interval $[0, \varepsilon)$ and $\varphi'(0^+) = a$. Let us consider a positively oriented base $a, b, v_1, \ldots, v_{n-2}$. Then the rotation in the plane $\{p_r, q_r\}$ is well defined by the condition that $q_r, p_r, v_1, \ldots, v_{n-2}$ is positively oriented. Thus $\Phi(x) = A(r)x$, $r = \|x\|$. The matrix $A(r)$ depends on cosine of the angle between the vectors q_r, p_r and, consequently, it is a C^∞-function for $r \geqslant 0$.

We put $g(x) = f(\Phi(x))$. Then

$$\operatorname{grad} g(rb) \| b, \quad \operatorname{grad} g(rb) \neq 0 \quad \text{for } r \in [0, \varepsilon).$$

Indeed, each point $x \in V \cap S_r$ is a critical point of the restriction $f|_{S_r}: S_r \to S_r$ and, consequently, by (6) the restriction $g|_{S_r}$ has a critical point at each $x = rb$, $r \in [0, \varepsilon)$. Consequently,

(7) $$\operatorname{grad} g(rb) = \sigma(r) b, \quad \sigma(r) \neq 0, \ r \in (0, \varepsilon).$$

Let us introduce a new parametrization $r = r(a)$ such that

(8) $$\operatorname{grad} g(r(a)b) = r'(a) b.$$

Equality (8) implies $r'(a) = \sigma(r(a))$, which is solvable for $a \in R$. Therefore, a curve $\tilde{\gamma}: a \to r(a)b$ is a trajectory of $\operatorname{grad} g$.

The local 1-parameter groups defined by $\operatorname{grad} g$ and $\operatorname{grad} f$ are conjugate by diffeomorphism Φ. Hence $\Phi^{-1}(\tilde{\gamma})$ is a trajectory of $\operatorname{grad} f$ with properties (1). This proves the Theorem.

Remark. Let $f: R^2 \to R$ be a C^∞-function with $\operatorname{grad} f(x) = 0$ for $x \in C$, where C is the logarithmic spiral $r = e^\varrho$. Then $\operatorname{grad} f(0) = 0$ and there exists no C^1-curve $\gamma: [0, \varepsilon) \to R^2$, $\gamma(0) = 0$, $\gamma \subset R^2 \setminus C$, with properties (1) when $x = 0$.

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