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REMARKS ON THE SPACE OF MONOTONIC FUNCTIONS

BY

JAN MYCIELSKI (BOULDER, COLORADO)

1. Introduction and motivations. Let M be the space of continuous nondecreasing functions $f\colon I\to I$ with $f(0)=0$ and $f(1)=1$, where $I=[0,1]$. With the distance $\max_{x\in I}|f_1(x)-f_2(x)|$, M is a complete metric space without isolated points. For any $f\in M$ we define its length by the natural formula

$$L(f)=\lim_{n\to\infty}\sum_{k=0}^{n-1}\left\{\left[f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right]^2+\frac{1}{n^2}\right\}^{1/2}.$$

Banach announced in [1], in a slightly different form (for a proof see [6]), that the set of all $f\in M$ with $L(f)<2$ is *meager* (i.e. of the first category). It is easy to see that the set of all $f\in M$ which are not strictly increasing is also meager. Therefore, the set of all $f\in M$ which are strictly increasing and satisfy $L(f)=2$ is comeager in M.

Let H be the space of continuous strictly increasing functions $f\colon I\to I$ with $f(0)=0$ and $f(1)=1$. We introduce in H the natural metrization

$$\max_{x\in I}|f_1(x)-f_2(x)|+\max_{x\in I}|f_1^{-1}(x)-f_2^{-1}(x)|,$$

which turns H into a complete metric group relative to composition of functions.

Again one can prove (by an easy modification of the argument of [6]) that the set of all $f\in H$ with $L(f)=2$ is comeager in H.

Now, it is well known that category and measure may disagree. For example, the set of numbers of I in whose binary developments 1 has frequency $\frac{1}{2}$ has measure 1 (Borel's strong law of large numbers). But the set of numbers of I in whose binary developments 1 has any frequency is meager (see [2], p. 100, for a nice game theoretic proof of S. Ulam of this fact). Therefore, we should ask if the above theorems of Banach are true or false in the measure theoretic sense. Unfortunately, neither the space M nor H carries a natural probability measure. But, as we have argued in [4], every natural space of analysis has a natural finite counterpart in which we have the obvious counting measure. And we shall now prove that the measure theoretic or probabilistic counterparts of the above theorems of Banach are false. We shall not write about length (because length is a very unstable functional, see [3]) and will show instead that the majority of nondecreasing functions cluster around the identity function. This supports Laplace's principle of insufficient reason (see Section 3).

We let

$$I_m = \{k/m \mid k = 0, \ldots, m\},$$

$$F_{mn} = \{f \mid f\colon I_m \to I_n,\ f \text{ is nondecreasing, } f(0) = 0, \text{ and } f(1) = 1\},$$

$$F_{m\infty} = \{f \mid f\colon I_m \to I,\ f \text{ is nondecreasing, } f(0) = 0, \text{ and } f(1) = 1\}.$$

Then F_{mn} has the uniform "counting" probability measure. As for $F_{m\infty}$, it can be identified with the $(m-1)$-dimensional simplex

$$\{(y_1, \ldots, y_m) \mid y_1 + \ldots + y_m = 1,\ y_i \geqslant 0\}$$

using the map $f \to (y_1, \ldots, y_m)$ defined by

$$y_i = f\left(\frac{i}{m}\right) - f\left(\frac{i-1}{m}\right).$$

Thus $F_{m\infty}$ gets the normalized $(m-1)$-dimensional Lebesgue measure from this simplex.

For all $m < \infty$ and $n \leqslant \infty$ we write $P(F_{mn}, \varepsilon)$ for the probability that an $f \in F_{mn}$ satisfies $|f(x) - x| < \varepsilon$ for all $x \in I_m$.

THEOREM. *For every* $\varepsilon > 0$ *we have*

(i) $$\lim_{m \to \infty} P(F_{m\infty}, \varepsilon) = 1,$$

(ii) $$\lim_{m \to \infty} P(F_{mn}, \varepsilon) = 1 \quad \textit{for } n+1 \geqslant 2/\varepsilon,$$

and

(iii) $$\lim_{m,n \to \infty} P(F_{mn}, \varepsilon) = 1,$$

(iv) $$\lim_{n \to \infty} P(F_{mn}, \varepsilon) = P(F_{m\infty}, \varepsilon).$$

(Statement (iii) was announced in [5].)

2. Proofs. First we need the following preliminaries.

Let X_{mk} $(m = 1, 2, \ldots, 0 \leqslant k \leqslant m)$ be random variables with values in I and distributions

$$\Pr(X_{mk} \leqslant a) = (m+1)\binom{m}{k} \int_0^a x^k (1-x)^{m-k}\, dx$$

(these are called *β-distributions*). As well known, and readily checked,

$$\text{(1)} \qquad \mathrm{E}\,(X_{mk}) = \frac{k+1}{m+2}$$

(this is the Bayesian estimator of the probability when one got k successes in m trials) and

$$\text{(2)} \qquad \mathrm{Var}\,(X_{mk}) = \frac{(k+1)(m-k+1)}{(m+2)^2(m+3)}.$$

Let

$$S(m, k, p, \varepsilon) = (m+1)\binom{m}{k} \int_{\max(0,p-\varepsilon)}^{\min(1,p+\varepsilon)} x^k (1-x)^{m-k}\, dx.$$

LEMMA. *If* $m \to \infty$, $k/m \to p$, *and* $\varepsilon > 0$, *then*

$$S(m, k, p, \varepsilon) \to 1.$$

Proof. By (1), $\mathrm{E}\,(X_{mk}) \to p$ and, by (2), $\mathrm{Var}\,(X_{mk}) \to 0$. Hence the Lemma follows.

Proof of the Theorem. (i) Let Δ_x^d denote the regular d-dimensional simplex of height x in R^d, and $|\Delta_x^d|$ its d-dimensional Lebesgue measure. Thus $|\Delta_x^d| = c_d x^d$, where c_d is a constant. Let $P(m, k, p, \varepsilon)$ be the probability that an $f \in F_{m\gamma}$ satisfies $|f(k/m) - p| < \varepsilon$. Thus, by our previous definition of the probability measure in $F_{m\gamma}$, we have

$$P(m, k, p, \varepsilon) = \frac{\int_{\max(0,p-\varepsilon)}^{\min(1,p+\varepsilon)} |\Delta_x^{k-1}|\,|\Delta_{1-x}^{m-k-1}|\, dx}{\int_0^1 |\Delta_x^{k-1}|\,|\Delta_{1-x}^{m-k-1}|\, dx} = S(m-2, k-1, p, \varepsilon).$$

We choose an integer $N \geqslant 6/\varepsilon$ and integers $0 = k_0(m) < \ldots < k_N(m) = m$ for each m such that $k_i(m)/m \to i/N$ for $i = 0, \ldots, N$. Therefore, by the Lemma,

$$\lim_{m\to\infty} \prod_{i=0}^{N} P\left(m, k_i(m), \frac{i}{N}, \frac{\varepsilon}{6}\right) = 1.$$

Hence to prove (i) it is enough to show that, for m large enough, the inequalities

$$\left| f\left(\frac{k_i(m)}{m}\right) - \frac{i}{N} \right| < \frac{\varepsilon}{6} \qquad \text{for } i = 0, \ldots, N$$

imply $|f(x) - x| < \varepsilon$ for all $x \in I_m$.

We choose $i < N$ such that

$$\frac{k_i(m)}{m} \leqslant x \leqslant \frac{k_{i+1}(m)}{m}.$$

Then, since $k_i(m)/m \to i/N$ and $k_{i+1}(m)/m \to (i+1)/N$, for m large enough we have $|x-i/N| < 2/N$. And since f is nondecreasing and $N \geqslant 6/\varepsilon$, we have

$$\begin{aligned} |f(x)-x| &\leqslant \left|f(x)-f\left(\frac{k_i(m)}{m}\right)\right| + \left|f\left(\frac{k_i(m)}{m}\right)-\frac{i}{N}\right| + \left|\frac{i}{N}-x\right| \\ &< \left|f\left(\frac{k_{i+1}(m)}{m}\right)-f\left(\frac{k_i(m)}{m}\right)\right| + \frac{\varepsilon}{6}+\frac{2}{N} \\ &\leqslant \left|f\left(\frac{k_{i+1}(m)}{m}\right)-\frac{i+1}{N}\right| + \frac{1}{N} + \left|\frac{i}{N}-f\left(\frac{k_i(m)}{m}\right)\right| + \frac{\varepsilon}{6}+\frac{2}{N} \\ &< \frac{\varepsilon}{2}+\frac{3}{N} \leqslant \varepsilon. \end{aligned}$$

(ii) and (iii). For all $y \in I$ we define

$$\psi_n(y) = \frac{k}{n} \text{ iff } \frac{k}{n+1} \leqslant y < \frac{k+1}{n+1}, \quad \text{and} \quad \psi_n(1) = 1.$$

We also define $\varphi_n\colon F_{m'} \to F_{mn}$ by $\varphi_n(f) = \psi_n \circ f$. Then it is easy to check that φ_n^{-1} is measure preserving and

$$|\varphi_n(f)(x)-f(x)| < \frac{1}{n+1}.$$

Hence, if $1/(n+1) \leqslant \varepsilon/2$, then

$$\varphi_n\{f \in F_{m'} \mid \forall x\, |f(x)-x| < \varepsilon/2\} \subseteq \{f \in F_{mn} \mid \forall x\, |f(x)-x| < \varepsilon\}.$$

Therefore

$$P(F_{m'}, \varepsilon/2) \leqslant P(F_{mn}, \varepsilon) \quad \text{for } n+1 \geqslant 2/\varepsilon.$$

Thus (ii) and (iii) follow from (i).

(iv) This is visible from well-known properties of the Lebesgue measure and the fact that the set $\{f \in F_{m'} \mid \forall x\, |f(x)-x| < \varepsilon\}$ is an open polyhedron in the simplex $F_{m'}$.

3. Conclusion. Laplace's principle of insufficient reason tells that if X is a random variable taking values in [0, 1] and we have no information about its distribution, then it is reasonable to assume that X is uniformly distributed. Since $F_{m'}$ is the space of distributions over $I_m - \{0\}$, our Theorem (i) supports this principle.

In any quantitative study of the physical reality which involves the space M of Section 1 it should be possible to replace M by a space F_{mn} with sufficiently large m and n. (Using the theory FIN of [4] we could also apply potentially infinite m and n; then a notion of continuity of such functions is also available, see [4].) Thus our theorem has more physical meaning than the theorem of Banach stated in Section 1.

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MATHEMATICS DEPARTMENT
UNIVERSITY OF COLORADO
BOULDER, COLORADO

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