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ON INDEPENDENT SEQUENCES IN TOPOLOGICAL LINEAR SPACES

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Throughout X denotes a Hausdorff topological linear space over the reals R. Following [7], Definition 1, we call a sequence (x_n) in X *topologically linearly independent*, resp., *topologically linearly m-independent*, if for each sequence $(\lambda_n) \in R^N$, resp., $(\lambda_n) \in m = l_\infty$, such that $\sum_{n=1}^{\infty} \lambda_n x_n = 0$ we have $(\lambda_n) = 0$. The first notion is also known under the name of ω-independence (cf. [5]).

The purpose of this note is to prove the following

THEOREM. *Assume (x_n) is a linearly independent sequence in X. Then there exists a topologically linearly independent subsequence (x_{n_k}) provided one of the following two conditions holds*:

(a) *(x_n) is a Cauchy sequence which does not converge to* 0.

(b) *X admits a continuous (homogeneous) norm* $\|\cdot\|$.

The proof is based on some results from [3], [7] and [8]. We quote here one of them in a slightly improved form which will be needed in the sequel.

PROPOSITION ([7], Proposition 1). *Assume (x_n) is a linearly independent sequence in X such that $\sum_{n=1}^{\infty} x_n$ is bounded multiplier Cauchy. Then there exists a subsequence (x_{n_k}) with $n_1 = 1$ which is topologically linearly m-independent.*

The argument given in [7] requires only a slight modification in order to yield the additional condition $n_1 = 1$ (cf. also the passage preceding Proposition 2 in [7]).

Proof of the theorem. (a) We may assume $x_n \to x$ and $x \neq 0$ (replacing X by its completion). We may also assume $x \notin \operatorname{lin}\{x_n : n \in N\}$ (omitting a finite number of x_n's). Finally, we may assume X is metrizable

([7], Lemma 3). Then, as the sequence

$$x,\ x_1-x,\ x_2-x,\ \ldots$$

is linearly independent and converges to 0, there exist $n_1 < n_2 < \ldots$ such that $\sum_{k=1}^{\infty} (x_{n_k}-x)$ is bounded multiplier Cauchy and the sequence

$$x,\ x_{n_1}-x,\ x_{n_2}-x,\ \ldots$$

is topologically linearly m-independent ([7], Lemma 4, and the Proposition above). Hence, by [3], Lemma, (x_{n_k}) is topologically linearly independent.

(b) We may assume $(X, \|\cdot\|)$ is separable and $\|x_n\| = 1$ for $n \in N$. Let (x_i^*) be a sequence of norm-continuous linear functionals on X such that $\|x\| = \sup\{|x_i^*(x)|: i \in N\}$ for all $x \in X$. Denote by ϱ the linear topology on X defined by the seminorms $x \to |x_i^*(x)|$, $i \in N$. Since $|x_i^*(x_n)| \leqslant 1$ for all $i, n \in N$, using the diagonal procedure, we can find a subsequence (y_n) of (x_n) which is Cauchy in (X, ϱ). If $y_n \not\to 0$ in (X, ϱ), it is enough to apply the proved part of the Theorem. In the opposite case, by a result of C. Bessaga (see [8], Proposition; cf. also the generalizations due to Kalton [6], Theorem 3.2, and Drewnowski [2], Theorem 2.2), (y_n) contains a basic subsequence.

Remarks. 1. In case X is complete, metrizable and locally convex (i.e., a B_0-space), condition (b) holds if and only if X contains no subspace isomorphic to R^N (see [1]).

2. The second assertion of the Theorem appears, in equivalent form, in [4] and [5]. However, as claimed in [5], the proof given in [4] is invalid. Unfortunately, there is a gap in [5], too. Namely, Theorem 1 of [5] fails in every infinite-dimensional (Banach) space $(X, \|\cdot\|)$. Indeed, given $x_0 \in X$ and a sequence (z_n) in X such that $z_n \to x_0$ and $x_0 \notin \operatorname{lin}\{z_n: n \in N\}$, we can define inductively a subsequence (x_n) with $x_1 = z_1$ and

$$\Big\|x_0 - \sum_{k=1}^{n} \alpha_k x_k\Big\| > \sum_{k=n+1}^{\infty} \|x_0 - x_k\|$$

for all $\alpha_1, \ldots, \alpha_k \in R$; $n \in N$. Taking $z_1 = 0$, we see that (x_n) need not even be linearly independent.

Nevertheless, the proof given above bears some resemblance to that of [5].

Postscript. Recently monograph [9] came to the author's attention. It contains an example, due to V. M. Kadec, of a linearly independent sequence in R^N without topologically linearly independent subsequences (see p. 858). It also contains a correction of the Erdös–Straus proof [4], due to P. Terenzi, and announces yet another proof of the Erdös–Straus result, also due to Terenzi (ibidem, p. 858).

Added in proof. 1. As noted by L. Drewnowski, the proof of part (b) can entirely be based upon the proposition above and its version given in

Z. Lipecki and P. Terenzi, *Subsequences of independent sequences in topological linear spaces*, Accademia Nazionale delle Scienze dei XL, Rendiconti, Memorie di Matematica, 9 (1985), p. 25–31

(see Proposition 1). Indeed, since $\|x_n\| = 1$, the convergence of the series $\sum_{n=1}^{\infty} \lambda_n x_n$ implies $(\lambda_n) \in m$.

2. Our proof of part (b) yields, in fact, the following stronger result:

THEOREM′. *Let X be a normed space and let (x_n) be a linearly independent sequence in X. Then there exists a subsequence (x_{n_k}) which is topologically linearly independent with respect to the weak topology of X.*

That the latter independence is, indeed, stronger is seen from the following example: $X = l_2$ and $x_n = e_n - e_{n-1}$, where $e_0 = 0$ and (e_n) is the standard basis of l_2. Then $\sum_{n=1}^{\infty} x_n = 0$ weakly, but (x_n) is topologically linearly independent in $(X, \|\cdot\|)$.

3. In connection with the postscript see also

P. Terenzi, *Proof of a theorem on ω-independence in Banach spaces*, Istituto Lombardo, Accademia di Scienze e Lettere, Rendiconti A, 114 (1980), p. 56–64.

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