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EXPANDING STARS

BY

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All graphs considered here* are finite, undirected, and without loops or multiple lines. Let $S(n)$ denote an *n-star* graph which has $n+1$ points, where one point s has degree $\deg s = n$ and every other point has degree one. For $n \geqslant 2$, suppose we permit the lines of $S(n)$ to expand such that the growth of a line is stopped only when its endpoint of degree one meets the interior of any line, or meets any other point, and the resulting simplicial complex is realized by a graph G. Then we say that G is an *expanding n-star*. It may be emphasized that in the expanding process, the interior of a line of $S(n)$ can only be met by an endpoint. For given n, let $E(n)$ denote the collection of all non-homeomorphic graphs obtained by expanding $S(n)$. We note that it is possible for a graph to belong to both $E(n)$ and $E(m)$ for $n \neq m$.

In the literature we find references to $E(2)$ in seemingly diverse situations. Thus, Doyle [3] proved that every monotone union of 1-cells must be homeomorphic to one of the graphs in Fig. 1. Also, five of these configurations represent the termination of a self-avoiding walk discussed by Kesten [5]. More recently, Lelek and McAuley [7] proved that if K is a locally connected and locally compact metric space which is a one-to-one continuous image of the line, then K is homeomorphic to one of the graphs (a) through (e) of Fig. 1. In fact, these very considerations led to the introduction of the concept of expanding stars in [4], where some of the preliminary results were obtained.

Alternatively, a graph G is an *expanding n-star*, $n \geqslant 2$, if G can be written as $\bigcup_{i=1}^{\infty} S_i(n)$, where $S_1(n)$ is an n-star, $S_i(n)$ is homeomorphic to an n-star for $i \geqslant 2$, and $S_i(n) \subseteq S_{i+1}(n)$ for $i = 1, 2, \ldots$ It will be convenient to call the point s in $S_1(n)$, with $\deg s = n$, a *source* of the expanding n-star G. Obviously, an expanding star graph has not necessarily a unique source. So, if $G \in E(n)$, then there exists a continuous mapping $f\colon S(n) \to G$

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which is one-to-one except perhaps at the endpoints of $S(n)$. Using this, or otherwise, it is easy to show that $E(2)$ consists of the six graphs of Fig. 1. By a direct method, it was possible to obtain in [4] all the 30 elements of $E(3)$, and these are reproduced in Fig. 2.

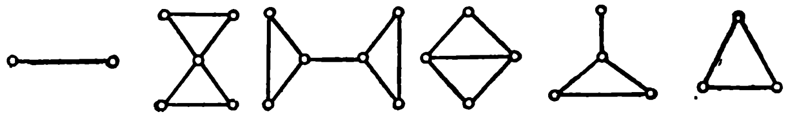

Fig. 1. The collection $E(2)$

For a graph G, let $D(k)$ represent the number of points of G whose degree is not less than k. As usual, $[x]$ denotes the greatest integer not greater than x.

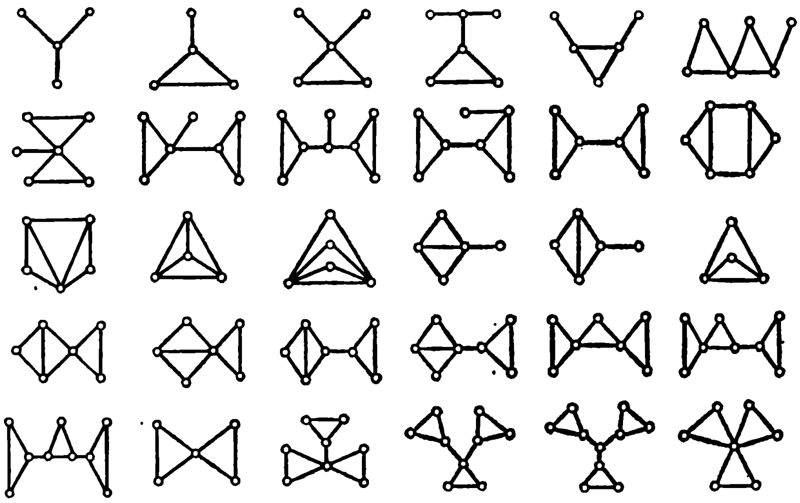

Fig. 2. The collection $E(3)$

THEOREM 1. *If* $G \in E(n)$, *then* $D(k) \leqslant 1+[n/(k-2)]$ *for* $3 \leqslant k \leqslant 2n$ *and* $n \geqslant 3$.

Proof. Let s be a source of G. Then $n \leqslant \deg s \leqslant 2n$. Starting from the n-star $S(n)$, the maximum number of points in G with degree not less than 3 is obtained, if, for example, the endpoint e of every line l in $S(n)$ meets l in its interior. Thus, $D(3) \leqslant 1+[n/1]$. In general, $k-2$ of the n endpoints of $S(n)$ have to meet on the interior of a line to produce a point of degree k in G. Since the n endpoints can be paired to produce at most $[n/(k-2)]$ points with degree k, we have the desired result.

The preceding theorem shows that we cannot have too many points of high degree in an expanding n-star G. For instance, if $n+3 \leqslant k \leqslant 2n$, then $D(k) \leqslant 1$; and $k = n+1$ or $n+2$, or $k = n \geqslant 5$ implies $D(k) \leqslant 2$.

In fact, all members of $E(n)$ are locally euclidean everywhere except at $n+1$ points at most. On the other hand, Theorem 9 shows that the scarcity of points of high degree is compensated by the existence of points of small degree. Also, if v is any point of an expanding n-star G, then $1 \leqslant \deg v \leqslant 2n$; and if $\deg v > n+2$, then v must be a unique source of G. Next, we directly obtain the following result from Theorem 1:

COROLLARY 2. *If a graph G is an expanding n-star, $n \geqslant 3$, and $\Delta(G)$ is the maximum degree of the points of G, then $(k-2)\big(D(k)-1\big) \leqslant n \leqslant \Delta(G)$ for $3 \leqslant k \leqslant 2n$.*

Using the current terminology of graph theory, let K_p denote the *complete* graph on p points where each points is adjacent to every other point; and let $K_{m,n}$ be the *complete bipartite* graph whose point set can be partitioned into subsets V_1 and V_2 having m and n elements, respectively, and where distinct points u and v are adjacent if and only if $u \in V_i$, $v \in V_j$ and $i \neq j$. Previous remarks and results show that graphs homeomorphic to $K_2, K_3, K_4, K_{1,n}$ and $K_{2,n}$ are each homeomorphic to some expanding t-star for suitable choices of t.

THEOREM 3. *The graphs K_5 and $K_{3,3}$ are not expanding n-stars for any $n \geqslant 2$.*

Proof. Expanding n-stars are completely classified for $n = 2$ and 3 in Figs. 1 and 2, respectively, and these do not contain K_5 or $K_{3,3}$. Consequently, $K_{3,3}$ fails to be an expanding n-star, since $n = 2$ or 3 are the only possibilities in this case. For K_5, then, $n = 4$ is the only candidate, and $\Delta(G) = k = 4$ yields a contradiction to the conclusion of Corollary 2.

Next, we will show that no expanding n-star graph can contain subgraphs homeomorphic to K_5 or $K_{3,3}$, and as in Kuratowski [6], this will prove that every graph in $E(n)$ is *planar*. In order to do this, we obtain a much stronger result, namely, if $G \in E(n)$ and H is any subgraph of G which does not contain a source of G, then all subgraphs of H are excluded from being homeomorphic to K_4 or $K_{2,3}$, and following Chartrand and Harary [2], this proves that H is *outerplanar*. To this end we proceed as follows.

In $S(n)$, let s be the source, and $v_1, v_2, \ldots, v_n$ — the endpoints of the n-lines. Let $S(n)$ expand to yield a graph G, with the line sv_i of $S(n)$ being transformed into a subgraph P_i of G, where P_i is a path or a cycle. Denote the $p_i \geqslant 0$ points on P_i (other than s and v_i) by $u_1, u_2, \ldots, u_{p_i}$, and note that together with s and v_i, these are the only points where the lines of $S(n)$ meet P_i with their endpoints. In $S(n)$, it is convenient to assign an orientation *s to v_i* for each i, indicated $s \to v_i$, and let this induce the natural orientations on the lines $su_1, u_1u_2, \ldots, u_{p_i}v_i$ of P_i (see Fig. 3). In what follows, an orientation will always refer to that due to a particular source which will be clear from the context.

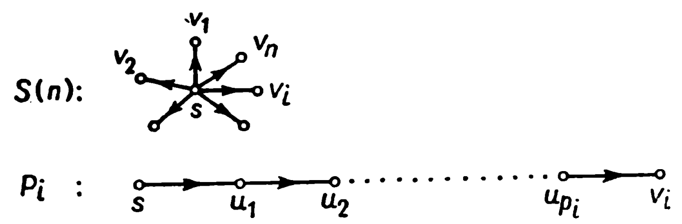

Fig. 3. An orientation

LEMMA 4. *Let s be a source of an expanding star G, and let v be any point of G other than s. Then at most one of the lines incident with v can have an orientation leading away from v.*

Proof. Assuming to the contrary, at least two of the expanding lines of $S(n)$ will have their interiors meeting at v.

LEMMA 5. *For $t \geqslant 3$, let $a_1, a_2, \ldots, a_t$ be the points of a path P in an expanding n-star G. If a source s of G does not belong to P, and if the line $a_{t-1}a_t$ has the orientation $a_{t-1} \to a_t$ due to s, then the lines $a_i a_{i+1}$ have an orientation $a_i \to a_{i+1}$ for $i = 1, 2, \ldots, t-2$.*

Proof. Lemma 4 can be successively applied to the points a_{t-1}, $a_{t-2}, \ldots, a_1$ to obtain the desired result (see Fig. 4).

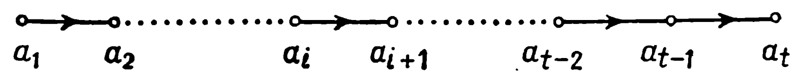

Fig. 4. An induced orientation on a path

A graph G is said to be *outerplanar* if it can be embedded in the plane so that every point of G lies on the exterior region. Chartrand and Harary [2] have characterized outerplanar graphs as those graphs which fail to contain subgraphs homeomorphic from K_4 or $K_{2,3}$.

THEOREM 6. *Let G be an expanding n-star graph with a source s. Let H be any subgraph of G not containing s. Then H is outerplanar.*

Proof. According to the characterization of outerplanar graphs stated above, we have to show that no subgraph of H is homeomorphic to K_4 or $K_{2,3}$. Assume to the contrary. Then, it is easy to see that there exists a subgraph A of H which is homeomorphic to the graph $K_4 - x$ (see Fig. 5).

Let a and b be the points of A with degree 3, every other point having degree 2. Consider the orientation in A due to the given source s. By

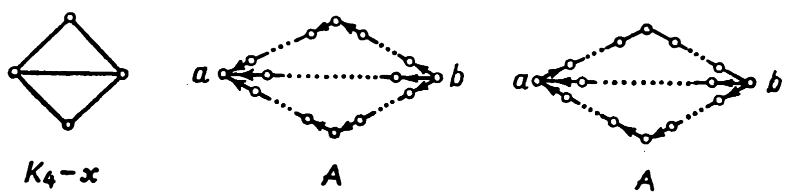

Fig. 5. Homeomorphs of $K_4 - x$

Lemma 4, there exist two possibilities for orientations along lines incident with a: either all are towards a or exactly one leads away from a. But then Lemma 5 applied to the paths in A between a and b shows that at b there are at least 2 lines leading away from that point. This contradicts Lemma 4.

COROLLARY 7. *If s is a source of an expanding star graph G, then the graph $G-s$ is outerplanar.*

THEOREM 8. *If a graph G is an expanding star, then G is planar.*

Proof. Let s be a source of G. Since the graph $G-s$ is outerplanar by Corollary 7, it can be embedded in the plane so that every point of this graph lies on the exterior region. Now G can be reconstructed from $G-s$ without disturbing planarity.

THEOREM 9. *Let a graph G on $p \geqslant 5$ points be an expanding star. Then G has* (i) *at least three points of degree at most four, and* (ii) *at least four points of degree at most five.*

Proof. If s is a source of G, then by a result in [1] and Corollary 7, the outerplanar graph $G-s$ has at least three points of degree at most three. In case any of these points is adjacent to s, it has degree not greater than 4 in G. This completes (i), and part (ii) follows directly from a similar result in [1], since the graph G is planar by Theorem 8.

The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors needed to color the points of G such that adjacent points are colored differently.

THEOREM 10. *Let a graph G be an expanding star. Then $\chi(G) \leqslant 4$.*

Proof. If s is a source of G, then $G-s$ is outerplanar. So $\chi(G-s) \leqslant 3$ (see [1]). Thus, at most three colors are needed for $G-s$, and, if necessary, a fourth color can be assigned to s in a minimal coloring of G.

In [1], a graph G was defined to have property P_n, $n \geqslant 1$, if G contains no subgraph which is homeomorphic from K_{n+1} or $K_{[(n+2)/2],\{(n+2)/2\}}$, where $\{x\} = -[-x]$. Thus, for $n = 1, 2, 3$ and 4, the property P_n is, respectively, *totally disconnected, forest, outerplanar* and *planar*. In view of the preceding theorem, it is of interest to investigate the embeddability of graphs with property P_n in expanding stars. The case $n = 4$ is of particular significance, for if every planar graph embeds in some expanding star, the Four Color Conjecture stands proved. Unfortunately, the embedding property breaks down even at $n = 3$ as we see in the following.

It is obvious that a totally disconnected graph G consisting of p isolated points (i. e., having property P_1) can be embedded in an expanding star by considering a suitable star graph. That the graph G_1 of Fig. 6 is not an expanding star shows that not every tree need be an expanding star. However, an induction argument can be used to prove that every tree T can be embedded in some expanding star G; indeed, we can further assure that T does not contain a source of G. So, if F is any forest (i. e., F has property P_2), then F can be embedded in some expanding star.

It was proved in Theorem 6 that if G is an expanding star with source s and H is any subgraph of G such that $s \notin H$, then H fails to contain any subgraph which is homeomorphic from the graph G_2 of Fig. 6. So, we observe that the outerplanar graph G_3 of Fig. 6 cannot be embedded in

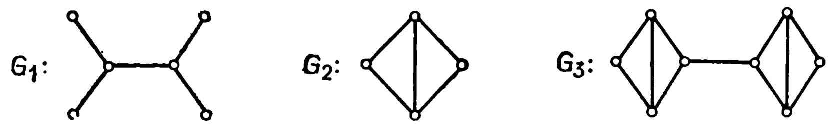

Fig. 6. Counterexamples on embedding

any expanding star. This settles in the negative the embedding question discussed above for all properties P_n, $n \geqslant 3$.

We have seen that expanding stars generate a large class of planar graphs whose chromatic number does not exceed four, and many interesting questions can now be posed. For example, we conclude by stating a combinatorial problem:

Is there any neat formula to determine the number of elements in $E(n)$? (**P 822**)

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