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# EXTREME POSITIVE OPERATORS ON INVOLUTION ALGEBRAS

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**Introduction.** In 1963 Phelps [23] has shown that the extreme positive unital operators defined on a selfadjoint algebra of bounded functions and taking values in a function algebra are exactly the multiplicative positive operators. This extends a result of A. and C. Ionescu-Tulcea [15], who proved an analogous assertion for the algebra of all continuous functions on a compact Hausdorff space. The methods of proof were quite algebraic and rather elementary; as indicated in [4] they can be applied to certain (not necessarily associative) ordered algebras (for definition see Section 1). A characterization of extreme positive *functionals* as the multiplicative positive functionals was given by several authors; for example by Tate [30] for a selfadjoint algebra of bounded functions, by R. Kadison (see [19]) for an ordered algebra with order unit, and by Bucy and Maltese [6] for a commutative unital Banach *-algebra. Generalizations to extreme positive operators on certain Banach *-algebras can be found in [12] and [17]. A treatment of extreme positive operators on ordered algebras was given by Donner [9] in 1976.

One aim of this paper is to show that the characterization of extreme positive operators on ordered algebras applied to complex *-algebras yields not only a more transparent treatment but also new and more general results.

In the first section we present a brief discussion of extreme positive operators on ordered algebras, since these characterizations are fundamental in the sequel. In the second section we discuss extreme positive functionals on certain *-algebras. As an application we obtain a surprisingly simple proof of a commutativity criterion given by Ogasawara [22] for a $C^*$-algebra and extend his criterion to LMC algebras with continuous involution and with a closed cone.

It is well known that every unital selfadjoint algebra of complex-valued functions on a set $X$ (separating the points of $X$) can be considered as a unital commutative LMC algebra with continuous involution when endowed with the topology of pointwise convergence. The cone of all pointwise positive functions on $X$ coincides with the closure of the wedge induced by the involution. In the third section we show that not every *-algebra is order isomorphic to a suitable function algebra. For example, a commutative Banach *-algebra $A$ is order isomorphic with a selfadjoint function algebra iff the wedge $A_+$ is closed and antisymmetric.

In the fourth section we discuss extreme positive operators on a unital commutative *-algebra taking values in a commutative LMC algebra with continuous involution. We denote the set of all positive operators satisfying $T1 \leqslant 1$ by $K_0(A, B)$, and the subset of all unital operators in $K_0(A, B)$ by $K_1(A, B)$. If the closed wedge of $B$ is antisymmetric and the algebraic unit element of $A$ is an order unit, then the extreme points of $K_i(A, B)$ are exactly the multiplicative operators in $K_i(A, B)$ $(i = 0, 1)$. One may replace the assumption of order unit by the hypothesis that the wedge $A_+$ is of type 0; for the definition see Section 1. For example, it is a direct consequence that the Arens algebra $L^\omega[0, 1]$ does not have an extreme unital positive functional.

In the last section we apply these results to unital commutative Banach *-algebras $A$, $B$ and obtain the results of [12] as an easy corollary. Moreover, we show that the symmetry of the involution of $B$ is a necessary condition in order that $K_i(A, B)$ be the closed convex hull of its extreme points in the strong operator topology under the additional assumption that $A$ has at least two unital positive functionals.

**1. Extreme positive operators on ordered algebras.** In this section we consider not necessarily associative algebras with unit over the field $\boldsymbol{R}$ or $\boldsymbol{C}$. A convex subset $A_+$ satisfying $\lambda A_+ \subset A_+$ for all $\lambda \in \boldsymbol{R}$ with $\lambda \geqslant 0$ is called a *wedge*. Every wedge $A_+$ induces a reflexive and transitive relation $\leqslant$, where $a \leqslant b$ is defined by $b - a \in A_+$. This order relation is compatible with the *real* vector space structure. A wedge $A_+$ is *antisymmetric* iff $0 \leqslant a \leqslant 0$ implies $a = 0$ or, equivalently, $A_+ \cap -A_+ = \{0\}$. For $e \in A_+$ we define the *order interval*

$$[0, e] := \{a \in A\colon\ 0 \leqslant a \leqslant e\}.$$

An element $e \in A_+$ is called an *order unit* if the linear span of $[0, e]$ is the entire vector space $A$, i.e., $\operatorname{Lin}[0, e] = A$. We say that $A$ is an *ordered algebra* if $A$ is a unital not necessarily associative algebra endowed with a wedge $A_+$ which contains the unit element 1 and is closed under multiplication, i.e., $a, b \geqslant 0$ implies $ab \geqslant 0$. A wedge $A_+$ is called *Archimedean* iff for $a \in A$, $b \in A_+$ the inequality $na \leqslant b$ for all $n \in N$ implies $a \leqslant 0$.

Now, let $A$, $B$ be unital algebras endowed with a wedge $A_+$, resp. $B_+$ (containing the unit element). An operator $T\colon A \to B$ is *positive* iff $T(A_+) \subset B_+$ and *unital* iff $T1 = 1$. As in [4], [16], [23] we define

$$L(A, B) := \{T\colon A \to B\colon\ T \text{ linear}\},$$

$$K_0(A, B) := \{T \in L(A, B)\colon\ T \text{ positive and } T1 \leqslant 1\},$$

$$K_1(A, B) := \{T \in L(A, B)\colon\ T \text{ positive and } T1 = 1\}.$$

A linear operator $T$ is called *Jordan multiplicative* if $T$ satisfies the relation $T(a^2) = (Ta)^2$ for all $a \in A$. For a subset $M \subset L(A, B)$ we denote the set of all multiplicative (resp., Jordan multiplicative) operators in $M$ by $\operatorname{Hom} M$ (resp., $\operatorname{J Hom} M$).

**1.1.** THEOREM. *Let $A$, $B$ be ordered algebras and $T \in K_i(A, B)$ an extreme point $(i = 0, 1)$. Then for all $a_0 \in \operatorname{Lin}[0, 1]$ and for all $a \in A$ we have the equalities*

$$T(a_0 a) = T(a_0)T(a) \quad \text{and} \quad T(aa_0) = T(a)T(a_0).$$

*In particular, if $1_A$ is an order unit, then*

$$\operatorname{ext} K_i(A, B) \subset \operatorname{Hom} K_i(A, B) \quad \text{for } i = 0, 1.$$

Proof. Let $0 \leqslant a_0 \leqslant 1$, $T \in K_i(A, B)$ and define

$$S_{a_0}(a) := T(a_0 a) - T(a_0)\, T(a),$$

which is a linear operator. Using the fact that $A_+$ and $B_+$ are closed under multiplication, it is easy to see that $T \pm S_{a_0}$ is in $K_i(A, B)$. Since $T$ is extreme, $S_{a_0} = 0$, i.e.,

$$T(a_0 a) = T(a_0)T(a).$$

Since $T$ is linear, this holds for all $a_0$ in the linear span of the order interval $[0, 1]$. Similarly the second equality follows. The proof is complete.

Our proof depends on the fact that the order interval $[0, 1]$ contains enough elements. In [4], Theorem 8, an example is given of an extreme positive operator defined on a *function algebra* which is not multiplicative. In that example the order interval consists only of constant functions. The assumption that the unit element is an order unit is very strong, but it seems essential. If the operators involved are continuous with respect to a certain topology, it is of course sufficient to assume that $\operatorname{Lin}[0, 1]$ is dense in $A$, i.e., 1 is a quasi-order unit (cf. [27], p. 241). Nevertheless, Bonsall, Lindenstrauss and Phelps have shown another way to guarantee that the order interval $[0, 1]$ contains enough elements: a wedge $A_+$ is called of *type* $n \in N \cup \{0\}$ iff $a \in A_+$ implies that $(1+a)^{-1}$ exists and $a^n(1+a)^{-1} \in A_+$.

Now, let $A$ be an ordered associative algebra and $A_+$ of type 0. For $a_0 \in A_+$ we have $0 \leqslant a_0 \leqslant 1 + a_0$. Since $A_+$ is closed under multiplication and $(1+a_0)^{-1} \in A_+$, we obtain

$$0 \leqslant a_0(1+a_0)^{-1} \leqslant 1.$$

If $T$ is extreme in $K_i(A, B)$ $(i = 0, 1)$, by Theorem 1.1 we obtain

$$T\big(a_0(1+a_0)^{-1}a\big) = T\big(a_0(1+a_0)^{-1}\big)\, Ta$$

and

$$T\big(aa_0(1+a_0)^{-1}\big) = Ta\,T\big(a_0(1+a_0)^{-1}\big)$$

for all $a \in A$. By using the associative law, a quite algebraic but tricky computation shows that these equations imply

$$T(a_0 a) = T(a_0) Ta$$

(cf. [4], p. 164). If the wedge $A_+$ generates $A$, i.e., Lin $A_+ = A$, we can conclude by the linearity of $T$ that

$$T(ba) = TbTa \quad \text{for all } a, b \in A.$$

So we have proved the following

**1.2.** THEOREM. *Let $A$, $B$ be ordered associative algebras and suppose that $A_+$ generates $A$ and is of type* 0. *Then*

$$\operatorname{ext} K_i(A, B) \subset \operatorname{Hom} K_i(A, B) \quad \textit{for } i = 0, 1.$$

EXAMPLE. Let $A$ be the algebra $M_n(C)$ of all $(n \times n)$-matrices over $C$. We say that a matrix is *positive* iff every coefficient is non-negative. It is obvious that $A$ is an ordered algebra relative to the product of matrices. The set $K_1(A, C)$ has extreme points (for example, the evaluation of the first diagonal element) but there is no multiplicative functional (relative to the product of matrices), i.e., we do not have

$$\operatorname{ext} K_1(A, C) \subset \operatorname{Hom} K_1(A, C).$$

Clearly, the unit element is not an order unit and the wedge $A_+$ is not of type 0.

Now, let us consider the converse implication, i.e.,

$$\operatorname{Hom} K_i(A, B) \subset \operatorname{ext} K_i(A, B).$$

If $A = B$, the identity operator is in $\operatorname{Hom} K_i(A, A)$. Then the validity of the above inclusion would imply the existence of an extreme point in $K_i(A, A)$. But it is not difficult to show that there is no extreme point in $K_i(A, B)$ if the wedge $A_+$ does not generate $A$ or if the wedge $B_+$ is not antisymmetric (cf. [23], p. 267, or [9]), so these conditions are necessary to prove the above inclusion.

Our proof rests on the Schwarz inequality. A different proof can be found in [9], which uses the fact that the unit element of $A$ is an order unit.

We call a positive operator $T: A \to B$ a *Schwarz map* iff

$$(Ta)^2 \leqslant T1T(a^2) \quad \text{for all } a \in A_+.$$

The following theorem is rather technical in its hypotheses — this hopefully will be justified by the applications to follow.

**1.3.** THEOREM. *Let $A$, $B$ be not necessarily associative unital algebras endowed with the wedge $A_+$, resp. $B_+$, containing the unit element of $A$, resp. $B$. Assume that $A_+$ generates $A$ and $B_+$ is antisymmetric and that every operator $T \in K_i(A, B)$ $(i = 0, 1)$ is a Schwarz map. Suppose furthermore that the square of the difference of two positive elements of $B$ is positive and that $(b_1 - b_2)^2 = 0$ implies $b_1 - b_2 = 0$ for all $b_1, b_2 \in B_+$. Then for $i = 0, 1$*

$$\mathrm{J}\operatorname{Hom} K_i(A, B) \subset \operatorname{ext} K_i(A, B).$$

Proof. Let $T$ be a Jordan homomorphism and let $T=\frac{1}{2}(T_1+T_2)$ be a convex combination of positive operators $T_1$, $T_2 \in K_i(A, B)(i=0, 1)$. The Schwarz inequality applied to $T_1$, $T_2$ yields for $a \in A_+$

$$\begin{aligned}\tfrac{1}{2}\left((T_1a)^2+(T_2a)^2\right) &\leqslant \tfrac{1}{2}T_1 1 T_1(a^2)+\tfrac{1}{2}T_2 1 T_2(a^2) \leqslant \tfrac{1}{2}(T_1(a^2)+T_2(a^2))\\ &= T(a^2) = (Ta)^2 = \tfrac{1}{4}[(T_1a)^2+T_1aT_2a+T_2aT_1a+(T_2a)^2].\end{aligned}$$

This yields $0 \geqslant \frac{1}{4}(T_1a-T_2a)^2 \geqslant 0$, where the last inequality follows from our hypothesis. Since $B_+$ is antisymmetric, we have $(T_1a-T_2a)^2=0$ and, by assumption, $T_1a=T_2a$. Since $A_+$ generates the algebra, $T_1=T_2$, i.e., $T$ is extreme.

**1.4.** Remark. It is not difficult to extend these results to the set $K_i'(A, B)$ of all continuous positive operators in $K_i(A, B)(i=0, 1)$ if $A$, $B$ are endowed with a vector space topology. Then under the assumptions of Theorem 1.3 we conclude that every positive multiplicative operator in $K_i'(A, B)$ is extreme. On the other hand, the operator $S_{a_0}$ is continuous if the multiplication is separately continuous. If we assume that the order interval $[0, 1]$ is total in $A$ and $A$, $B$ are ordered algebras, we conclude by Theorem 1.1 that an extreme operator is multiplicative.

A discussion of the extreme rays of the set of all positive operators can be found in [4], [9], [23], and in the case of an order complete vector space in [18].

**2. Extreme positive functionals on *-algebras.** Let $A$ be a unital complex *-algebra. We denote the set of all selfadjoint elements by $\operatorname{Sym} A$, i.e.,

$$\operatorname{Sym} A := \{a \in A:\ a=a^*\}.$$

It is obvious that the set

$$A_+ := \Big\{\sum_{i=1}^{N} a_i^* a_i:\ N \in \boldsymbol{N},\ a_i \in A,\ i=1, \ldots, N\Big\}$$

is a wedge. So we can apply the definitions of the first section. Since $\boldsymbol{C}$ is a *-algebra, a functional $f$ is positive if $f(A_+) \subset \boldsymbol{C}_+$ or, equivalently, $f(a^*a) \geqslant 0$ for all $a \in A$. The polarization formula shows that the wedge of a unital *-algebra is always generating (cf. [25]). We set $K_A := K_1(A, \boldsymbol{C})$.

**2.1.** Corollary. *Let $A$ be a unital *-algebra. Then every multiplicative positive unital functional is extreme in $K_A$.*

Proof. We apply Theorem 1.3. Obviously, it suffices to show that every positive functional is a Schwarz map, but this is an easy consequence of the Cauchy–Schwarz inequality.

The Jordan product $\circ$ of an algebra is defined by

$$a \circ b = \tfrac{1}{2}(ab+ba).$$

At first we will generalize a result of Bucy and Maltese [6]. If a *-algebra is endowed with a vector space topology, we denote the closure of $A_+$ by $\bar{A}_+$.

**2.2.** THEOREM. *Let $A$ be a unital Banach *-algebra. If $A_+$ or $\bar{A}_+$ is closed under Jordan multiplication, then* $\operatorname{ext} K_A = \operatorname{Hom} K_A$.

Proof. It is a consequence of the square root lemma of Ford that the unit element of a Banach *-algebra is an order unit for the wedge $A_+$ and, in particular, for $\bar{A}_+$ (see, e.g., [25]). Since $A$ is an ordered algebra relative to the Jordan product, we can apply Theorem 1.2.

**2.3.** PROPOSITION. *Let $A$ be an algebra with an Archimedean wedge $A_+$. Then the following assertions are equivalent*:

(a) *$a \geqslant 0$ and $b \geqslant 0$ imply $a \circ b \geqslant 0$.*

(b) *$0 \leqslant a \leqslant b$ implies $a \circ b \geqslant 0$.*

(c) *$0 \leqslant a \leqslant b$ implies $0 \leqslant a^2 \leqslant b^2$.*

Proof. The first implication is trivial and (b) $\Rightarrow$ (c) is easy, since $0 \leqslant a \leqslant b$ implies $0 \leqslant b-a \leqslant b+a$ and, by (b),

$$0 \leqslant (b-a) \circ (b+a) = (b^2 - a^2).$$

For the last implication consider for all $n \in N$ the element

$$c_n := a + \tfrac{1}{n} b \quad \text{with } a, b \geqslant 0.$$

Then $0 \leqslant a \leqslant c_n$ implies $0 \leqslant a^2 \leqslant c_n^2$, which yields $-n(ab+ba) \leqslant b^2$. Since $A_+$ is Archimedean, we obtain $ab+ba \geqslant 0$.

The statement (c) is called the *condition of Ogasawara* (see, e.g., [28], [31]). Ogasawara has shown in [22] that this condition implies the commutativity of a $C^*$-algebra. Since the wedge $A_+$ of a $C^*$-algebra is closed and antisymmetric, it is a consequence of

**2.4.** COROLLARY. *Let $A$ be a unital Banach *-algebra. Suppose that $\bar{A}_+$ is antisymmetric and the condition of Ogasawara holds for $\bar{A}_+$. Then $A$ is commutative.*

Proof. Since a closed wedge is Archimedean, we see by Proposition 2.3 that $\bar{A}_+$ is closed under Jordan multiplication. By Theorem 2.2 we know that every extreme point of $K_A$ is multiplicative. Since $K_A$ is the weak star closure of the convex hull of its extreme points, we have

$$f(i(uv - vu)) = 0 \quad \text{for all } f \in K_A \text{ and for all } u, v \in \operatorname{Sym} A.$$

But then $i(uv - vu)$ is in $\bar{A}_+ \cap -\bar{A}_+ = \{0\}$; see formula (1) below. Thus $A$ is commutative.

**2.5.** Remark. The wedge $A_+$ of a unital Banach *-algebra is Archimedean iff $A_+$ is closed. This stems from the fact that $A_+$ generates $A$ and the unit element is an interior point of $A_+$ (cf. [27], p. 222).

Now, let us consider more general *-algebras. If $A$ is a unital *-algebra endowed with a locally convex vector space topology, $P_A$ denotes the set of all unital positive continuous functionals on $A$, and $\tilde{P}_A$ the set of all unital positive functionals whose restrictions to $\operatorname{Sym} A$ are continuous. If the involution is continuous, then these sets coincide. Using a separation theorem for the real locally convex vector space $\mathrm{Sym} A$ it is not very difficult to see that

(1) $$a \in \bar{A}_+ \cap \mathrm{Sym} A \Leftrightarrow a \in \mathrm{Sym} A \text{ and } f(a) \geqslant 0 \text{ for all } f \in \tilde{P}_A$$

(cf. Proposition 1.1 in [25]). A *topological algebra* is a topological vector space with jointly continuous multiplication. An algebra with a locally convex vector space topology is called a *locally-m-convex algebra*, or an *LMC algebra* for short, iff there exists a family $(p_\alpha)_{\alpha \in I}$ of submultiplicative seminorms generating the topology of $A$. One can assume that the family contains the maximum of any finite number of seminorms of the family. We denote the completion of $A/p_\alpha^{-1}(\{0\})$ by $A_\alpha$ and the projection of $a \in A$ in $A_\alpha$ by $a_\alpha$. It is well known that an LMC algebra can be topologically embedded into the projective limit of the Banach algebras $A_\alpha$. A complete metrizable LMC algebra is called a *Fréchet algebra*.

For a unital LMC algebra with continuous involution we denote the set of all unital positive functionals which are bounded for a given seminorm $p_\alpha$ by $P_\alpha$. Clearly, then

$$P_A = \bigcup_{\alpha \in I} P_\alpha.$$

Brooks [5] has shown that $P_A$ is the w*-closed convex hull of its extreme points and

$$\mathrm{ext} P_A = \bigcup_{\alpha \in I} \mathrm{ext} P_\alpha.$$

Furthermore, $P_\alpha$ is affinely homeomorphic to $P_{A_\alpha}$, in particular

$$\mathrm{ext} P_\alpha = \mathrm{ext} P_{A_\alpha}.$$

It is easily seen that the closure of the wedge $(A_\alpha)_+$ in the Banach *-algebra $A_\alpha$ is closed under Jordan multiplication if $A_+$ or $\bar{A}_+$ are closed under Jordan multiplication. This gives

**2.6.** COROLLARY. *Let $A$ be a unital LMC algebra with continuous involution. If $A_+$ or $\bar{A}_+$ are closed under Jordan multiplication, then* $\mathrm{ext} P_A = \mathrm{Hom} P_A$. *If the condition of Ogasawara holds for the closed cone $\bar{A}_+$, then $A$ is commutative.*

**2.7.** LEMMA. *Let $A$ be a unital *-algebra and $B$ be a unital commutative LMC algebra with continuous involution or a unital commutative Banach *-algebra. Then every positive map $T: A \to B$ is a Schwarz map and the inequality*

$$(Ta)^* Ta \leqslant T1\, T(a^* a)$$

*holds for all $a \in A$ relative to $\bar{B}_+$.*

Proof. Since $T$ is positive, the element

$$b := T1\, T(a^* a) - (Ta)^* Ta$$

is selfadjoint. Because $P_A$ is the w*-closed convex hull of its extreme points, it suffices to show by (1) that $f(b) \geqslant 0$ for all $f \in \operatorname{ext} P_A$. Since an extreme functional is multiplicative, we have

$$f(b) = (f \circ T(1))(f \circ T(a^* a)) - |f \circ T(a)|^2.$$

But $f \circ T$ is a positive functional and the statement follows now from the Cauchy–Schwarz inequality.

**3. Function algebras and *-algebras.** Let $A$ be a unital algebra of complex-valued functions defined on a set $X$. We assume that $A$ separates the points of $X$. In the case of real-valued functions one may consider the complexification. A function $a \in A$ is called *positive* iff $a(x) \geqslant 0$ for all $x \in X$. We denote the set of all positive functions by $A_X$ and the complex conjugate function by $\bar{a}$. The algebra is *selfadjoint* iff $a \in A$ implies $\bar{a} \in A$. It is evident that $A$ is selfadjoint iff the cone $A_X$ generates $A$.

As mentioned above the existence of an extreme positive operator on $A$ implies that the wedge $A_X$ generates the algebra (cf. Proposition 7.2 of [23]). So we will assume that $A$ is selfadjoint. Then $A$ is an LMC algebra with a continuous involution, where the involution is given by complex conjugation and the topology is induced by the submultiplicative seminorms $p_x$ $(x \in X)$ defined by $p_x(a) := |a(x)|$, i.e., the topology of pointwise convergence. Furthermore, the relation $A_+ \subset A_X$ is obvious. But, in general, this inclusion is proper. For example, consider the algebra of all polynomials over $\boldsymbol{C}$ regarded as functions on the interval $[0, 1]$. Then the polynomial $1 - z$ is in $A_X$, but not in $A_+$, since $A_+$ contains only polynomials of even degree. Thus the ordering induced by the set $X$ is different from the ordering induced by the involution. It is an interesting fact that we have $A_+ = A_X$ if we consider the polynomials as functions on the line $\boldsymbol{R}$.

If $A_+$ and $A_X$ are different, the term positive functional is ambiguous. To avoid confusion we define by $K_{A_X}$ the set of all unital functionals on $A$ which are positive relative to $A_X$ and similarly $K_{A_+}$. For every $x \in X$ the *Dirac functional* $\delta_x$ is defined by

$$\delta_x(a) := a(x) \quad \text{with } a \in A.$$

The inclusion $\{\delta_x \colon x \in X\} \subset K_{A_X} \subset K_{A_+}$ is trivial. Corollary 2.1 yields

$$\{\delta_x \colon x \in X\} \subset \operatorname{Hom} K_{A_+} \subset \operatorname{ext} K_{A_+}.$$

But even in the case $A_+ = A_X$ this inclusion may be proper. For example, consider the algebra of all continuous bounded functions on a completely regular Hausdorff space. In this case one can identify $\operatorname{ext} K_{A_+}$ with the Stone–Čech compactification of $X$. Note, however, that $\{\delta_x \colon x \in X\}$ is dense in $\operatorname{ext} K_{A_+}$.

Although the next theorem may be well known we will sketch the proof since we do not know a reference for it.

**3.1.** THEOREM. *Let $A$ be a unital selfadjoint algebra of complex-valued functions on a set $X$. Then $A_X$ is the closure of $A_+$ in the topology of pointwise convergence and $P_{A_+}$ is the* w*-*closed convex hull of the set $\{\delta_x : x \in X\}$.*

Proof. Let $a \in A_X$, i.e., $a(x) \geqslant 0$ for all $x \in X$. We consider the expansion of the square root function: for all $n \in N$ there exists $k_n \in N$ such that for all $\beta \in \boldsymbol{R}$ with $0 \leqslant \beta < 2$ we have

$$\left| \sum_{k=0}^{k_n} \binom{\frac{1}{2}}{k} (\beta - 1)^k - \sqrt{\beta} \right| \leqslant 1/n.$$

Define

$$a_n := \sqrt{n} \sum_{k=0}^{k_n} \binom{\frac{1}{2}}{k} \left( \frac{1}{n} a - 1 \right)^k.$$

Then $a_n$ is a selfadjoint element and we want to show that $a_n^2$ converges to $a$ in the topology of pointwise convergence. It suffices to show that $a_n(x)$ converges to $\sqrt{a(x)}$ for every $x \in X$. But for $n > a(x)$ we have

$$|a_n(x) - \sqrt{a(x)}| \leqslant \frac{1}{\sqrt{n}}.$$

For the second statement assume that $f_0$ is in $P_A$ but not in the w*-closed convex hull of $\{\delta_x : x \in X\}$ denoted by $C$. It is well known that we can separate $f_0$ and $C$ by a linear w*-continuous functional on $A'$. Since the dual of $(A', \text{w}^*)$ is isomorphic to $A$ via evaluation, we can find a selfadjoint element $a \in A$ such that

$$\hat{a}(f_0) = f_0(a) < \inf\{\hat{a}(y) : y \in C\} =: \alpha.$$

But then $a - \alpha$ is in $A_X$ but not in $\bar{A}_+$, a contradiction.

The next theorem is well known and is listed for completeness.

**3.2.** THEOREM. *Let $A$ be a selfadjoint algebra of bounded functions on a set $X$ endowed with the supremum norm $\| \ \|_\infty$. Then $A_X$ is the norm closure of $A_+$. If $A$ is complete, we have $A_+ = A_X$.*

Now it is natural to ask whether every *-algebra $A$ corresponds to a certain space of complex-valued functions. We say that $A$ is *order isomorphic* to a function space $F$ on a set $X$ iff there exists a vector space isomorphism $\varrho: A \to F$ which preserves the order in both directions, i.e., $a \in A_+$ iff $\varrho(a)(x) \geqslant 0$ for all $x \in X$. We will not require that the multiplication be preserved.

**3.3.** PROPOSITION. *Let $A$ be a unital Fréchet algebra with continuous involution or a unital Banach *-algebra. Then $A$ is order isomorphic to a function space $F$ iff the wedge $A_+$ is antisymmetric and closed.*

Proof. Since $F_X$ is antisymmetric, this carries over to $A_+$. For every $x \in X$ the functional $\delta_x \circ \varrho$ is positive, and therefore automatically continuous since $A$ is a Banach *-algebra, resp. a Fréchet algebra with continuous involution.

Since

$$A_+ = \bigcap_{x \in X} (\delta_x \circ \varrho)^{-1}([0, \infty)),$$

the wedge $A_+$ is closed. The converse is a consequence of the next proposition. Observe that

$$A_+ = \bar{A}_+ \cap \operatorname{Sym} A$$

if $A_+$ is closed.

3.4. PROPOSITION. *Let $A$ be a unital *-algebra with a locally convex vector space topology. Assume that $\bar{A}_+ \cap \operatorname{Sym} A$ is antisymmetric. Then*

$$a \in \bar{A}_+ \cap \operatorname{Sym} A \quad \textit{iff} \quad f(a) \geqslant 0 \textit{ for all } f \in \tilde{P}_A.$$

*Thus $(A, \bar{A}_+ \cap \operatorname{Sym} A)$ is order isomorphic to a function space.*

Proof. In view of (1) it suffices to show that $f(a) \geqslant 0$ for all $f \in \tilde{P}_A$ implies that $a$ is selfadjoint. Let $a = u + iv$ with $u, v \in \operatorname{Sym} A$. Since $f(u), f(v)$ are real, we have $f(v) = 0$ for all $f \in \tilde{P}_A$. By (1),

$$v \in \bar{A}_+ \cap -\bar{A}_+ \cap \operatorname{Sym} A = \{0\}.$$

For the second statement define $X := \tilde{P}_A$ and $\varrho(a): X \to \boldsymbol{C}$ by

$$\varrho(a)(f) := f(a) \quad (f \in \tilde{P}_A)$$

and $F$ as the image of $\varrho$. By the first statement, $\varrho$ preserves the order. $\varrho$ is injective since $\varrho(a) = 0$ implies, by the first statement, that

$$a \in \bar{A}_+ \cap -\bar{A}_+ \cap \operatorname{Sym} A = \{0\}.$$

We have given in [25] an example of a Banach *-algebra $A$ with a cone $A_+$ which is not closed.

**4. Extreme positive operators on commutative *-algebras.** Let $A$, $B$ be unital *-algebras with a vector space topology. We have seen in Theorems 3.1 and 3.2 that besides $A_+$ the wedge $\bar{A}_+$ is of great interest. Thus it is natural to consider linear operators $T: A \to B$ with the condition

$$T(A_+) \subset \bar{B}_+ \quad \text{or} \quad T(\bar{A}_+) \subset \bar{B}_+,$$

etc. The notation of the sets $K_i(A, B)$ does not indicate the corresponding wedge. In order to avoid a boundless notation we substitute the letters $A$, $B$ by the corresponding wedges. For example, $K_1(\bar{A}_+, B_+)$ denotes the set of all unital linear operators with $T(\bar{A}_+) \subset B_+$. The following inclusions are trivial $(i = 0, 1)$:

$$K_i(\bar{A}_+, B_+) \subset K_i(A_+, B_+) \subset K_i(A_+, \bar{B}_+) \quad \text{and} \quad K_i(\bar{A}_+, \bar{B}_+) \subset K_i(A_+, \bar{B}_+).$$

If we want to assert a statement simultaneously for the wedges $A_+$ and $\bar{A}_+$, we say that the statement holds with $A_* = A_+, \bar{A}_+$ using the letter $A_*$ as a variable for $A_+, \bar{A}_+$.

**4.1.** THEOREM. *Let $A$ be a unital *-algebra (endowed with a vector space topology in the case $A_* = \bar{A}_+$) and $B$ be a unital commutative LMC algebra with continuous involution. If $\bar{B}_+$ is antisymmetric, we have*

$$\operatorname{Hom} K_i(A_*, B_*) \subset \operatorname{ext} K_i(A_*, B_*)$$

*for $A_* = A_+, \bar{A}_+$ and $B_* = B_+, \bar{B}_+$, and $i = 0, 1$.*

Proof. We apply Theorem 1.3 for the case $B_* = \bar{B}_+$ and $A_* = A_+$. By Lemma 2.7 every positive operator is a Schwarz map. The condition that the square of the difference of two positive elements is positive is trivial for the wedge $B_+$. Since the multiplication is jointly continuous, this property carries over to $\bar{B}_+$. Now let $b_1, b_2 \in \bar{B}_+$ with $(b_1 - b_2)^2 = 0$. The Cauchy–Schwarz inequality shows that

$$f(b_1 - b_2) = 0 \qquad \text{for all } f \in P_A.$$

By (1) we have $b_1 - b_2 \in \bar{B}_+ \cap -\bar{B}_+$, which shows that $b_1 - b_2 = 0$. So we have proved that

$$\operatorname{Hom} K_i(A_+, \bar{B}_+) \subset \operatorname{ext} K_i(A_+, \bar{B}_+).$$

Now let $T$ be an element of $\operatorname{Hom} K_i(A_*, B_*)$. It is trivial that then $T$ is in $\operatorname{Hom} K_i(A_+, \bar{B}_+)$, and therefore extreme in $K_i(A_+, \bar{B}_+)$. But then $T$ is also extreme in the subset $K_i(A_*, B_*)$.

It is remarkable that the above inclusion even holds for a non-commutative $C^*$-algebra (cf. Theorem 3.5 of [29] and Theorem 3.6 of [32]). Indeed, the commutativity of $B$ was only used to conclude that a positive operator is a Schwarz map. But for $C^*$-algebras $A$, $B$ this is just the inequality of Kadison. For further results on non-commutative algebras, we refer to [2].

**4.2.** COROLLARY. *Let $A$ be a unital commutative *-algebra and $B$ be a unital commutative LMC algebra with continuous involution and $\bar{B}_+$ antisymmetric. If the wedge $A_+$ is of type 0 or the unit element is an order unit for $A_+$, then*

$$\operatorname{Hom} K_i(A_+, B_*) = \operatorname{ext} K_i(A_+, B_*)$$

*with $B_* = B_+, \bar{B}_+, i = 0, 1$.*

Proof. Since $A$ is commutative, the wedge $A_+$ is closed under multiplication, so $A$ is an ordered algebra, and similarly $B$. Since the multiplication of $B$ is jointly continuous, $\bar{B}_+$ is also closed under multiplication. Now apply Theorems 1.1, 1.2 and 4.1.

**4.3.** COROLLARY. *Let $A$ be a unital commutative topological *-algebra and $B$ be a unital commutative LMC algebra with continuous involution and $\bar{B}_+$ antisymmetric. If the wedge $\bar{A}_+$ is of type 0 or the unit element of $A$ is an order unit for $\bar{A}_+$, then*

$$\operatorname{Hom} K_i(\bar{A}_+, B_*) = \operatorname{ext} K_i(\bar{A}_+, B_*)$$

*with $B_* = B_+, \bar{B}_+, i = 0, 1$.*

This follows from Theorems 1.1, 1.2 and 4.1.

As an application we consider the Arens algebra $L^{\omega}[0, 1]$ (for the definition see [1], p. 96). Obviously, the wedge $A_+$ is of type 0. Thus the extreme positive functionals are multiplicative by Corollary 4.2. But the zero functional is the only multiplicative functional on the Arens algebra, which is proved in [1], p. 104. Thus $\operatorname{ext} K_{L^{\omega}} = \emptyset$.

The involution of a *-algebra $A$ is called *symmetric* if the spectrum $\sigma_A(a^*a)$ is non-negative for all $a \in A$.

4.4. Proposition. *Let $A$ be a complete unital LMC algebra with continuous involution. Then the wedge $\bar{A}_+$ is of type 0 iff the involution is symmetric.*

Proof. For all $a \in A$ we have $a^*a \in \bar{A}_+$. If the closed wedge is of type 0, we know that $(1+a^*a)^{-1}$ exists. It is well known that then the involution is symmetric (cf. Section 32 in [10]). Now assume that the involution is symmetric and let $a \in \bar{A}_+$. It is not very difficult to reduce the case to a unital Banach algebra with a symmetric continuous involution since $P_A$ is the union of the sets $P_\alpha$ and the spectrum $\sigma_A(a)$ is the union of the sets $\sigma_{A_\alpha}(a_\alpha)$ calculated in the Banach algebra $A_\alpha$ with $\alpha \in I$. Since $a$ is selfadjoint, there exists a maximal abelian selfadjoint subalgebra $M$ containing $a$. Then $M$ is a commutative Banach algebra with symmetric involution. Every multiplicative unital functional $f\colon M \to C$ is positive and can be extended as a positive unital functional on $A$ (cf. Theorem 5.10 7° and 6.4 in [24]). Since $a$ is positive in $A$, we obtain $f(1+a) \geq 1$. But then $1+a$ is invertible and positive.

The following theorem is an easy consequence of Remark 1.4.

4.5. Theorem. *Let $A$ be a unital commutative topological *-algebra and $B$ be a unital commutative LMC algebra with continuous involution and $\bar{B}_+$ antisymmetric. If the unit element of $A$ is a quasi-order unit for $A_+$ (resp. $\bar{A}_+$), then*

$$\operatorname{Hom} K_i'(A_*, B_*) = \operatorname{ext} K_i'(A_*, B_*)$$

*for $B_* = B_+, \bar{B}_+$ and $i = 0, 1$, and $A_* = A_+$ (resp. $A_* = \bar{A}_+$).*

4.6. Remark. We have assumed in the last theorems that the involution of the range space $B$ is continuous. If $B$ is a Banach *-algebra and $\bar{B}_+$ is antisymmetric, then the involution is automatically continuous. This is clear since $\bar{B}_+$ is antisymmetric iff $B$ is *-semisimple (cf. Theorems 4.2 and 4.4 in [24]). In particular, $B$ is semisimple, and thus the involution is continuous.

**5. Positive operators and norm conditions.** Let $A$, $B$ be unital algebras of complex-valued bounded functions. It is well known (Theorem 1.3 in [23]) that

(2) $$K_1(A, B) = \{T \in L(A, B)\colon\ T1 = 1,\ \|Ta\|_\infty \leq \|a\|_\infty \text{ for all } a \in A\}.$$

It is remarkable that the set on the left-hand side is defined by purely algebraic conditions while the set on the right-hand side contains a norm property.

For a *-algebra $A$ we define

$$|a|_{P_A} := \sup\{|f(a)|: f \in P_A\}.$$

5.1. PROPOSITION. *Let $A$, $B$ be unital algebras endowed with a locally convex vector space topology and with a continuous involution. If $\bar{B}_+$ is antisymmetric and $P_A$ is* w*-*bounded, then*

$$K_1(\bar{A}_+, \bar{B}_+) = \{T \in L(A, B): T1 = 1, |Ta|_{P_B} \leqslant |a|_{P_A}\}.$$

This result can be deduced from (2) by using the fact that $A$, $B$ are order isomorphic to a function space (cf. Proposition 3.4).

In the last section we will discuss the extreme points of a convex set introduced by Espelie [12]. If $A$, $B$ are unital Banach *-algebras, we define

$$S_+(A, B) := \{T \in L(A, B): T1 = 1, \|Ta\| \leqslant \|a\| \text{ for all } a \in A \text{ and } T \text{ positive}\},$$

$$U_+(A, B) := \{T \in L(A, B): \|Ta\| \leqslant \|a\| \text{ for all } a \in A \text{ and } T \text{ positive}\}.$$

Then the first of the inclusions

$$(3) \qquad S_+(A, B) \subset K_1(A, B) \quad \text{and} \quad U_+(A, B) \subset K_0(A, B)$$

is trivial. In the second case observe that $\|T1\| \leqslant 1$ implies that $\sqrt{(1-T1)}$ exists, i.e., $T1 \leqslant 1$ in $A_+$. Let $T \in K_0(A, B)$. If the involution is isometric, we have $|Ta|_{P_A} \leqslant \|a\|$ for all $a \in A$. If $B$ is a commutative $C^*$-algebra, then $\|Ta\| = |Ta|_{P_A}$, i.e., the inclusions (3) are equalities. Thus we obtain the following result of [12] (Theorem 3):

5.2. THEOREM. *Let $A$ be a unital commutative Banach algebra with isometric involution and $B$ be a unital commutative $C^*$-algebra. Then*

$$U_+(A, B) = K_0(A, B) \quad \textit{and} \quad S_+(A, B) = K_1(A, B).$$

*In particular, the extreme points are exactly the multiplicative operators.*

For the proof apply Corollary 4.2.

The closed wedge $\bar{B}_+$ of a semisimple Banach algebra with symmetric involution is antisymmetric (see, e.g., [24], Theorem 6.6). Thus the following is a slight improvement of Theorem 1 of [12].

5.3. THEOREM. *Let $A$, $B$ be unital commutative Banach algebras with isometric involution and $\bar{B}_+$ antisymmetric. Then*

$$\operatorname{Hom} U_+(A, B) \subset \operatorname{ext} U_+(A, B) \quad \textit{and} \quad \operatorname{Hom} S_+(A, B) \subset \operatorname{ext} S_+(A, B).$$

Proof. By Theorem 4.1 we have

$$\operatorname{ext} K_0(A, B) \supset \operatorname{Hom} K_0(A, B) \supset \operatorname{Hom} U_+(A, B).$$

Thus every multiplicative operator $T$ of $U_+(A, B)$ is extreme in $K_0(A, B)$. Therefore $T$ is extreme in the subset $U_+(A, B)$. The same argument applies to $S_+(A, B)$.

The disk algebra is an example where the inclusion of Theorem 5.3 is proper (cf. the example in [12], p. 62).

A linear operator $T$: $A \to B$ is called a *spectral contraction* if

$$|Ta|_\sigma \leqslant |a|_\sigma \quad \text{for all } a \in A,$$

where $| \ |_\sigma$ denotes the spectral radius.

**5.4.** PROPOSITION. *Let $A$, $B$ be unital commutative Banach *-algebras and $\bar{B}_+$ be antisymmetric. Then the closure with respect to the strong operator topology of the convex hull of the extreme points of $K_i(A, B)$ is contained in the set of all spectral contractions.*

Proof. By Corollaries 4.2 and 4.3 we know that every extreme positive operator is multiplicative, and thus a spectral contraction. This property carries over to convex combinations and to the closure in the strong operator topology. The proof is complete.

**5.5.** LEMMA. *Let $A$ be a unital Banach *-algebra. Then the order interval* $[0, 1]$ *is spectrally bounded iff the involution is symmetric.*

Proof. It is not very difficult to see that $A_+ \cap -A_+$ is spectrally bounded if and only if $\bar{A}_+ \cap -\bar{A}_+$ is spectrally bounded. For $a \in A$ it is well known that the selfadjoint element $|a^*a|_{P_A} - a^*a$ is contained in $\bar{A}_+$, i.e.,

$$0 \leqslant a^*a \leqslant |a^*a|_{P_A} =: m(a)^2,$$

for example see formula (1). Thus $[0, 1]$ is spectrally bounded iff $m(a)^2 \leqslant 1$ implies $|a^*a|_\sigma \leqslant C$ for some $C > 0$. This is equivalent to

$$|a^*a|_\sigma \leqslant Cm(a)^2.$$

By Theorem 6.5 of [24] this is equivalent to the symmetry of involution.

**5.6.** THEOREM. *Let $A$, $B$ be unital commutative Banach *-algebras and $\bar{B}_+$ be antisymmetric. If $K_1(A, B)$ (resp. $K_0(A, B)$) is the closed convex hull of its extreme points in the strong operator topology and $P_A$ contains at least two elements (resp. one), then the involution of $B$ is symmetric.*

Proof. Let $f_1 \neq f_2 \in P_A$. Then there exists a selfadjoint element $a_0 \in A$, linearly independent of the unit element, with $f_1(a_0) \neq f_2(a_0)$. By adding a constant and multiplying with a real scalar we can assume that $f_1(a_0) = 1$ and $f_2(a_0) = 0$. Let $b_0 \in [0, 1]$ and consider the unital positive operator

$$T := b_0 f_1 + (1 - b_0) f_2.$$

By Proposition 5.4, $T$ is a spectral contraction. In particular, we have $|Ta_0|_\sigma = |b_0|_\sigma \leqslant |a_0|_\sigma$. Thus the order interval is spectrally bounded. Now apply Lemma 5.5. In the second case consider $T := b_0 f_1$.

It is easy to see that $K_i(A, B)$ contains no elements if $P_A$ is void. Similarly, $K_1(A, B)$ consists of a single element if $P_A$ consists of a single element (with the general assumption that $\bar{B}_+$ is antisymmetric or, equivalently, $P_B$ separates the points of $B$).

If $B$ is $C^*$-equivalent, i.e., $B$ is topologically isomorphic to a $C^*$-algebra, then one may use a result of [21] as follows: by $\tilde{A}$ we denote the enveloping $C^*$-algebra of the unital commutative Banach $*$-algebra $A$ (cf. Theorems 4.5 and 4.6 of [24]). It is easy to see that $K_i(\tilde{A}, B)$ is affinely and topologically isomorphic to $K_i(A, B)$ endowed with the strong operator topology induced by $\tilde{A}$, resp. $A$.

Morris and Phelps [21] have shown that $K_i(\tilde{A}, B)$ is the convex hull of its extreme points iff $P_B$ is Stonean.

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