COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 1

# *CONCERNING TWO COVERING PROPERTIES*

BY

RASTISLAV TELGÁRSKY (BRATISLAVA)

In this contribution we shall consider the following properties of topological spaces:

1° a space $X$ has a *closure-preserving cover* of compact sets, i.e., there exists a family $\{C_i: i\in I\}$ of compact subsets of $X$ such that $\bigcup\{C_i: i\in I\} = X$ and $\bigcup\{C_i: i\in J\}$ is closed in $X$ for each $J\subset I$;

2° a space $X$ has a *hereditarily closure-preserving cover* of compact sets, i.e., there exists a family $\{C_i: i\in I\}$ of compact subsets of $X$ such that $\bigcup\{C_i: i\in I\} = X$ and $\overline{\bigcup\{D_i: i\in I\}} = \bigcup\{\overline{D}_i: i\in I\}$, where $D_i\subset C_i$ for each $i\in I$.

Clearly, 2° implies 1°.

Property 1° was introduced by Tamano [9] and it was further studied by Potoczny [6], [7] and [8] and by the author [11] and [12].

Each space considered in this paper is assumed to be completely regular and each map is assumed to be continuous. For the topological background we refer to Engelking [1].

First draft of this paper has been sent to Proceedings of Japan Academy in June 1971 and its results were quoted, e.g., by Morita [5]. However, as there was much delay in dealing with it on the part of Proceedings, the paper has been eventually withdrawn and this is a new version made up to date only.

THEOREM 1. *Each $\sigma$-compact space has a closure-preserving cover of compact sets.*

Proof. Let $X$ be $\sigma$-compact. Then $X = \bigcup\{C_n: n\geq 1\}$, where each $C_n$ is a compact subset of $X$. Let us set $C_n' = \bigcup\{C_k: k\leq n\}$. Then it is easy to verify that $\{C_n': n\geq 1\}$ is a closure-preserving cover of $X$ of compact sets.

THEOREM 2. *Each locally compact paracompact space has a hereditarily closure-preserving cover of compact sets.*

Proof. Let $X$ be locally compact and paracompact. Then $X$ has a locally finite closed cover consisting of compact sets. Since each locally finite family of sets is hereditarily closure-preserving, the theorem follows.

Let $E$ be a closed subset of $X$. By $E^{(1)}$ we denote the set of all points $x \in E$ such that $E \cap \overline{U}$ is not compact for any neighbourhood $U$ of $x$ in $X$. We set $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$ and $X^{(\alpha)} = \bigcap \{X^{(\beta)}: \beta < \alpha\}$ for limit ordinal $\alpha$. A space $X$ is said to be *C-scattered* if $X^{(\alpha)} = 0$ for some ordinal $\alpha$. $C$-scattered spaces were studied in [10] (cf. also [11] and [12]).

THEOREM 3. *Let $X$ be a $C$-scattered space such that $X - X^{(\alpha)}$ is paracompact for each ordinal $\alpha$. Then $X$ has a closure-preserving cover of compact sets.*

Hence, if $X$ is $C$-scattered and hereditarily paracompact, then $X$ has a closure-preserving cover of compact sets.

Before the proof of Theorem 3 we prove two lemmas.

LEMMA 1. *Let $X$ be a paracompact space such that for a locally compact closed subset $Z$ of $X$ the subspace $X - Z$ of $X$ has a closure-preserving cover of compact sets. Then $X$ has a closure-preserving cover of compact sets.*

Proof. Let $X$ be a paracompact space, let $Z$ be a locally compact closed subset of $X$, and let $\{C_i: i \in I\}$ be a closure-preserving cover of $X - Z$ consisting of compact sets. There exists a locally finite family $\{U_t: t \in T\}$ of open sets in $X$ such that $Z \subset \bigcup \{U_t: t \in T\}$ and $Z \cap \overline{U_t}$ is compact for each $t \in T$. We set $Y = X - \bigcup \{U_t: t \in T\}$ and $C_i' = C_i \cap Y$ for each $i \in I$. Then $Y$ is a closed subset of $X$ and $\{C_i': i \in I\}$ is a closure-preserving cover of $Y$ consisting of compact sets. We set $C_{t,i} = (\overline{U_t} \cap Z) \cup (\overline{U_t} \cap C_i)$, where $t \in T$ and $i \in I$. Clearly, each $C_{t,i}$ is compact. It remains to prove that the family

$$\mathscr{A} = \{C_i': i \in I\} \cup \{C_{t,i}: t \in T \text{ and } i \in I\}$$

is a closure-preserving cover of $X$. Clearly, $\mathscr{A}$ covers $X$. Let $\mathscr{A}' \subset \mathscr{A}$. Then there exist $I' \subset I$, $T' \subset T$ and $I_t \subset I$, where $t \in T'$, such that

$$\mathscr{A}' = \{C_i': i \in I'\} \cup \{C_{t,i}: t \in T' \text{ and } i \in I_t\}.$$

Clearly, $\bigcup \{C_i': i \in I'\}$ is a closed subset of $Y$ and thus it is closed in $X$. We set $D_t = \bigcup \{C_{t,i}: i \in I_t\}$ for each $t \in T'$. Since the family $\{D_t: t \in T'\}$ is locally finite in $X$, it is sufficient to point out that $D_t$ is closed in $X$ for each $t \in T'$. Let $t \in T'$. Since $\bigcup \{\overline{U_t} \cap C_i: i \in I_t\}$ is closed in $\overline{U_t} - Z$, it follows that its closure in $X$ is contained in

$$\bigcup \{\overline{U_t} \cap C_i: i \in I_t\} \cup (\overline{U_t} \cap Z).$$

Thus $\bigcup \{C_{t,i}: i \in I_t\}$ is closed in $X$. Therefore $\mathscr{A}$ is closure-preserving.

LEMMA 2. *Let $\{F_t: t \in T\}$ be a locally finite closed cover of $X$ and let $\{C_{t,i}: i \in I_t\}$ be a closure-preserving cover of $F_t$ consisting of compact sets* *for each $t \in T$. Then $\{C_{t,i}: i \in I_t, t \in T\}$ is a closure-preserving cover of $X$ consisting of compact sets.*

The proof being easy is omitted.

Proof of Theorem 3. We proceed by induction with respect to $\alpha$ such that $X^{(\alpha)} = 0$. If $\alpha = 0$, then $X = 0$ and the theorem is true. Let $\alpha = \beta + 1$. Then $X^{(\beta)}$ is a locally compact closed subset of $X$ and, by the inductive assumption, $X - X^{(\beta)}$ has a closure-preserving cover of compact sets. Hence the theorem follows from Lemma 1. Let $\alpha$ be a limit ordinal. Then $\{X - X^{(\beta)}: \beta < \alpha\}$ is an open cover of $X$. Let $\{F_t: t \in T\}$ be a locally finite closed refinement of $\{X - X^{(\beta)}: \beta < \alpha\}$. Then for each $t \in T$ there exists $\beta_t < \alpha$ such that $F_t^{(\beta_t)} = 0$. It follows from the inductive assumption that $F_t$ has a closure-preserving cover of compact sets for each $t \in T$ and the theorem follows from Lemma 2.

As a corollary to Theorem 3 we have

THEOREM 4. *Each C-scattered metrizable space has a closure-preserving cover of compact sets.*

For scattered spaces one can prove (in the same manner as for $C$-scattered spaces) the following

THEOREM 5. *Each hereditarily paracompact scattered space has a closure-preserving cover consisting of finite sets.*

THEOREM 6. *Let us assume that there exists a perfect map from a space $X$ onto a space $Y$. Then $X$ has a closure-preserving cover of compact sets if and only if $Y$ has a closure-preserving cover of compact sets.*

The theorem follows from Lemma 3 and Lemma 4.

LEMMA 3. *Let $f$ be a closed map from a space $X$ onto a space $Y$ and let $\{C_i: i \in I\}$ be a closure-preserving cover of $X$ consisting of compact sets. Then $\{f(C_i): i \in I\}$ is a closure-preserving cover of $Y$ consisting of compact sets.*

Indeed, $f(C_i)$ is compact, since $f$ is continuous. The family $\{f(C_i): i \in I\}$ is closure-preserving, because $f$ is a closed map and $f$ preserves unions (cf. [3]).

LEMMA 4. *Let $f$ be a perfect map from a space $X$ onto a space $Y$ and let $\{C_i: i \in I\}$ be a closure-preserving cover of $Y$ consisting of compact sets. Then $\{f^{-1}(C_i): i \in I\}$ is a closure-preserving cover of $X$ consisting of compact sets.*

Indeed, $f^{-1}(C_i)$ is compact (cf. [1], p. 167). The family $\{f^{-1}(C_i): i \in I\}$ is closure-preserving, because $f$ is continuous and $f^{-1}$ preserves unions.

THEOREM 7. *A space $X$ has a hereditarily closure-preserving cover of compact sets if and only if $X$ is a closed image of a locally compact paracompact space.*

Let us note that closed images of paracompact locally compact spaces were studied by Ishii [2] and Morita [4].

Theorem 7 follows from Theorem 2 and from Lemmas 5 and 6.

LEMMA 5. *Let $\{C_i\colon i \in I\}$ be a hereditarily closure-preserving cover of $X$ consisting of compact sets. Let us set*

$$X' = \bigcup \{C_i \times \{i\}\colon i \in I\} \subset X \times I,$$

*where $I$ is considered as a discrete space and $X'$ as a subspace of $X \times I$. Then the space $X'$ is paracompact and locally compact and the projection $f\colon X' \to X$, defined by setting $f(x, i) = x$, where $x \in C_i$ and $i \in I$, is continuous and closed.*

Proof. The space $X'$ is paracompact and locally compact because it is the free union of compact spaces. We claim that $f$ is continuous. Let $E \subset X$. Then

$$f^{-1}(E) = \{(x, i) \in X'\colon x \in E\} = \bigcup \{(E \cap C_i) \times \{i\}\colon i \in I\}.$$

Hence, if $E$ is a closed subset of $X$, then the set $f^{-1}(E)$ is closed in $X'$. We claim that $f$ is closed. Let $E$ be a closed subset of $X'$. Then $E \cap (C_i \times \{i\})$ is a closed subset of $C_i \times \{i\}$. Let us set $D_i = \{f(x, i)\colon (x, i) \in E\}$. Then $D_i$ is a closed subset of $C_i$, because $f|(C_i \times \{i\})$ is a homeomorphism. We have

$$\begin{aligned} f(E) &= f\big(\bigcup \{E \cap (C_i \times \{i\})\colon i \in I\}\big) = \bigcup \{f\big(E \cap (C_i \times \{i\})\big)\colon i \in I\} \\ &= \bigcup \{D_i\colon i \in I\}. \end{aligned}$$

Since the family $\{C_i\colon i \in I\}$ is hereditarily closure-preserving, it follows that the set $\bigcup \{D_i\colon i \in I\}$ is closed in $X$. Hence $f(E)$ is a closed subset of $X$.

LEMMA 6. *Let $f$ be a closed map from a space $X$ onto a space $Y$ and let $\{C_i\colon i \in I\}$ be a hereditarily closure-preserving cover of $X$ consisting of compact sets. Then $\{f(C_i)\colon i \in I\}$ is a hereditarily closure-preserving cover of $Y$ consisting of compact sets.*

Proof. Let $D_i$ be a closed subset of $f(C_i)$ and let $E_i = C_i \cap f^{-1}(D_i)$. Then $f(E_i) = D_i$. Since the family $\{C_i\colon i \in I\}$ is hereditarily closure-preserving and $E_i$ is a closed subset of $C_i$ for each $i \in I$, it follows that the set $\bigcup \{E_i\colon i \in I\}$ is closed in $X$. Since the map $f$ is closed, $f\big(\bigcup \{E_i\colon i \in I\}\big)$ is closed in $Y$. However,

$$f\big(\bigcup \{E_i\colon i \in I\}\big) = \bigcup \{f(E_i)\colon i \in I\} = \bigcup \{D_i\colon i \in I\}.$$

Hence $\bigcup \{D_i\colon i \in I\}$ is a closed subset of $Y$ and, therefore, the family $\{f(C_i)\colon i \in I\}$ is hereditarily closure-preserving.

THEOREM 8. *Let $\{C_i\colon i \in I\}$ be a hereditarily closure-preserving cover of a space $X$ consisting of compact sets. Then for each compact subset $C$ of $X$ there exists a finite subset $J$ of $I$ such that $C \subset \bigcup \{C_i\colon i \in J\}$.*

Proof. Let $X'$ and $f$ be the same as in Lemma 5 and let $C$ be a compact subset of $X$. Suppose that $C$ is not contained in the union of any finite subfamily of $\{C_i: i \in I\}$. Then there exist sequences $\{x_n: n \in N\} \subset C$ and $\{C_{i_n}: n \in N\} \subset \{C_i: i \in I\}$ with $x_1 \in C_{i_1}$ and $x_{n+1} \in C_{i_{n+1}} - \bigcup \{C_{i_k}: 1 \leqslant k \leqslant n\}$ for each $n \in N$. Since $\{(x_n, i_n): n \in N\}$ is a closed discrete set in $X'$ and the map $f$ is closed, the set $\{f(x_n, i_n): n \in N\} = \{x_{i_n}: n \in N\}$ is discrete and closed in $X$. It contradicts, however, the compactness of $C$.

THEOREM 9. *If a space $X$ has a hereditarily closure-preserving cover consisting of paracompact closed subsets of $X$, then $X$ is paracompact.*

Proof. Let $\{P_i: i \in I\}$ be a hereditarily closure-preserving cover of $X$, where $P_i$ is a paracompact closed subset of $X$ for each $i \in I$. Let $\mathscr{A}$ be an open cover of $X$. For each $i \in I$ there exists a locally finite family $\mathscr{A}_i$ of closed subsets of $X$ such that $\mathscr{A}_i$ refines $\mathscr{A}$ and $\bigcup \mathscr{A}_i = P_i$. It is easy to point out that the family $\bigcup \{\mathscr{A}_i: i \in I\}$ is closure-preserving. By the theorem of Michael [3], $X$ is paracompact if and only if each open cover of $X$ has a closure-preserving closed refinement. Hence the theorem follows.

## REFERENCES


[1] R. Engelking, *Outline of general topology*, Amsterdam 1968.

[2] T. Ishii, *On product spaces and product mappings*, Journal of the Mathematical Society of Japan 18 (1966), p. 166-181.

[3] E. Michael, *Another note on paracompact spaces*, Proceedings of the American Mathematical Society 8 (1957), p. 822-828.

[4] K. Morita, *On closed mappings*, Proceedings of the Japan Academy of Sciences 32 (1956), p. 539-543.

[5] — *Some results on M-spaces*, Colloquia Mathematica Societatis János Bolyai, 8. Topics in Topology, p. 489-503, North Holland Publ. Co., Amsterdam 1974.

[6] H. Potoczny, *A nonparacompact space which admits a closure-preserving cover of compact sets*, Proceedings of the American Mathematical Society 32 (1972), p. 309-311.

[7] — *On a problem of Tamano*, Fundamenta Mathematicae 75 (1972), p. 29-31.

[8] — *Closure-preserving families of compact sets*, General Topology and its Applications 3 (1973), p. 243-248.

[9] H. Tamano, *A characterization of paracompactness*, Fundamenta Mathematicae 72 (1971), p. 189-201.

[10] R. Telgársky, *C-scattered and paracompact spaces*, ibidem 73 (1971), p. 59-74.

[11] — *Closure-preserving covers*, ibidem 85 (1974), p. 165-175.

[12] — *Spaces defined by topological games*, ibidem 88 (1975), p. 193-223.


INSTITUTE OF MATHEMATICS
SLOVAK ACADEMY OF SCIENCES

*Reçu par la Rédaction le 30. 3. 1975*