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INFINITE GAMES AND SINGULAR SETS

BY

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1. Introduction. A generalization of the classical Mazur game is presented* in which the players are permitted to choose sets from any family of sets. The sets for which the players have winning strategies are characterized for some special families of sets; e.g., all perfect subsets of the unit interval, all closed subsets of the unit interval with positive Lebesgue measure, all subsets of the unit interval having order type $1+\lambda+1$, etc. For this characterization, the concept of a singular set is introduced and leads to a unified method of classifying "negligible sets" in mathematics, e.g. sets of the first category, sets of Lebesgue measure zero, etc.

If $\mathscr{C}$ is a non-empty family of subsets of a non-empty set X and S is a subset of X, then the game $\Gamma(S, \mathscr{C})$ is played as follows:

Two players, I and II, alternately choose sets in $\mathscr{C}$ to define a descending sequence of sets, player I selecting the sets in the sequence with odd index and player II selecting the sets with even index. If the intersection of the constructed sequence has at least one point in common with S, then player I wins; otherwise player II wins. When $X = [0, 1]$ and $\mathscr{C}$ is the family of all closed subintervals of X with non-empty interiors, then $\Gamma(S, \mathscr{C})$ is the Mazur game.

The problem is to characterize those sets S for which the players have winning strategies.

The solution to this problem for a topological generalization of the Mazur game has been given by Oxtoby [10]. In Section 2 the concept of an "$\mathfrak{M}$-family" is introduced and provides a somewhat different generalization of the Mazur game. Using the terminology and proofs of Oxtoby,

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with slight modification, the solution to the problem when $\mathscr{C}$ is an $\mathfrak{M}$-family is presented in Section 3. Several examples of $\mathfrak{M}$-families are presented in Section 4 and some problems are given in Section 5.

2. Definitions. Let $\mathscr{E}$ be a family of subsets of a set X. A member of $\mathscr{E}$ will be called an *$\mathscr{E}$-set*.

Definition. A family $\mathscr{C}$ of subsets of a non-empty set X is called an *$\mathfrak{M}$-family* if $\mathscr{C}$ satisfies the following axioms:

1. The intersection of any descending sequence of $\mathscr{C}$-sets is non-empty.

2. Suppose x is a point in X. Then

a. there is a $\mathscr{C}$-set containing x, i.e. $X = \bigcup \mathscr{C}$; and

b. for each $\mathscr{C}$-set A, there is a $\mathscr{C}$-set $B \subseteq A$ such that $x \notin B$.

3. Let A be a $\mathscr{C}$-set and let $\mathscr{D}$ be a non-empty family of disjoint $\mathscr{C}$-sets which has the power less than the power of $\mathscr{C}$.

a. If $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then there is a $\mathscr{D}$-set D such that $A \cap D$ contains a $\mathscr{C}$-set.

b. If $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set, then there is a $\mathscr{C}$-set $B \subseteq A$ which is disjoint from all sets in $\mathscr{D}$.

Remark. The family of all closed intervals $[a, b]$ with $0 \leqslant a < b \leqslant 1$, the family of all closed sets with positive Lebesgue measure, and the family of all perfect sets are $\mathfrak{M}$-families of subsets of the unit interval. Additional examples are given in Section 4.

The solution of the problem when $\mathscr{C}$ is an $\mathfrak{M}$-family depends upon the following classification of sets which is a generalization of Baire category:

Definitions. Let $\mathscr{C}$ be a family of subsets of X. A subset S of X is ***singular with respect to $\mathscr{C}$***, or, more briefly, ***$\mathscr{C}$-singular***, if each $\mathscr{C}$-set A contains a $\mathscr{C}$-set B disjoint from S. A countable union of $\mathscr{C}$-singular sets is called a ***$\mathscr{C}_{\mathrm{I}}$-set***. A set which is not expressible as a countable union of $\mathscr{C}$-singular sets is called a ***$\mathscr{C}_{\mathrm{II}}$-set***. A subset S of X is said to have ***the Baire property with respect to $\mathscr{C}$*** if there is no $\mathscr{C}$-set A such that, for every $\mathscr{C}$-set $B \subseteq A$, both $B \cap S$ and $B \cap (X - S)$ are $\mathscr{C}_{\mathrm{II}}$-sets.

Game-theoretical definitions. θ will denote the empty sequence. A play is a descending sequence $\langle A_n \rangle_{n=1}^{\infty}$ of $\mathscr{C}$-sets, and the result of the play is the set $\bigcap_{n=1}^{\infty} A_n$. A strategy for player I is a function σ defined for all finite sequences of even length in $\mathscr{C}$ and the empty sequence such that $\sigma(\theta)$ is a $\mathscr{C}$-set and, for $n \geqslant 1$, $\sigma(A_1, \ldots, A_{2n})$ is a $\mathscr{C}$-set contained in A_{2n}. A strategy for player II is a function τ defined for all finite sequences of odd length in $\mathscr{C}$ such that, for each $n \geqslant 1$, $\tau(A_1, \ldots, A_{2n-1})$ is a $\mathscr{C}$-set contained in A_{2n-1}. To each strategy σ for player I and each strategy τ for player II there is associated a play $\langle\sigma, \tau\rangle$ defined inductively as follows: $A_1 = \sigma(\theta)$, and $A_{2n} = \tau(A_1, \ldots, A_{2n-1})$ and $A_{2n+1} = \sigma(A_1, \ldots, A_{2n})$ for each $n \geqslant 1$. If σ and τ are strategies for players I and II, respectively, then a play $\langle A_n\rangle_{n=1}^{\infty}$ is *consistent with* σ if $A_1 = \sigma(\theta)$ and $A_{2n+1} = \sigma(A_1, \ldots, A_{2n})$ for $n \geqslant 1$; *consistent with* τ if $A_{2n} = \tau(A_1, \ldots, A_{2n-1})$ for $n \geqslant 1$; and *consistent with* σ *and* τ if it is consistent with both σ and τ.

Let S be a subset of X. A strategy σ for player I is winning for player I in the game $\Gamma(S, \mathscr{C})$ if, for every strategy τ of player II, the result of the play $\langle\sigma, \tau\rangle$ intersects S. A strategy τ for player II is winning for player II in the game $\Gamma(S, \mathscr{C})$ if, for every strategy σ of player I, the result of the play $\langle\sigma, \tau\rangle$ does not intersect S. The game $\Gamma(S, \mathscr{C})$ is *determined* if either player I or player II has a winning strategy.

3. The Mazur game for $\mathfrak{M}$-families. Lemmas 1 and 2 will be used to prove Theorem 1, which characterizes the sets for which player II has a winning strategy.

LEMMA 1. *Each $\mathfrak{M}$-family $\mathscr{N}$ of subsets of X contains a maximal subfamily $\mathscr{M}$ of disjoint sets such that $X - \bigcup\mathscr{M}$ is $\mathscr{N}$-singular.*

Proof. Let Λ be the smallest ordinal number whose power is the same as the power of $\mathscr{N}$ and let $N_0, N_1, \ldots, N_\alpha, \ldots$ be a well-ordering of $\mathscr{N}$ into a transfinite sequence of type Λ. Set $M_0 = N_0$ and assume M_β has been defined for all $\beta < \alpha$, where $\alpha < \Lambda$. If $\{M_\beta \mid \beta < \alpha\}$ is a maximal family of disjoint sets, then set $M_\alpha = M_0$. Otherwise, let $N_0^\alpha, N_1^\alpha, \ldots$ be all those members of $\mathscr{N}$ whose intersection with $\bigcup\limits_{\beta<\alpha} M_\beta$ contains no $\mathscr{N}$-set, and define M_α to be the first subset of N_0^α which is an $\mathscr{N}$-set and is disjoint from all M_β for $\beta < \alpha$ (the well-ordering $N_0^\alpha, N_1^\alpha, \ldots$ being the restriction of the well-ordering of $\mathscr{N}$). Then $\mathscr{M} = \{M_\alpha \mid \alpha < \Lambda\}$ is the desired maximal family. Set $U = \bigcup\mathscr{M}$. It will now be shown that $X - U$ is $\mathscr{N}$-singular. Assume N is an $\mathscr{N}$-set, say $N = N_\alpha$, where $\alpha < \Lambda$. If $N \cap (\bigcup\limits_{\beta<\alpha} M_\beta)$ contains an $\mathscr{N}$-set, then there is an $\mathscr{N}$-set disjoint from $X - U$. If $N \cap (\bigcup\limits_{\beta<\alpha} M_\beta)$ does not contain an $\mathscr{N}$-set, then there is an $\mathscr{N}$-set contained in N which is disjoint from all M_β, with $\beta < \alpha$, and M_α is the first $\mathscr{N}$-set contained in N_0^α which is disjoint from all M_β for $\beta < \alpha$. But, by virtue of the construction employed, $N_0^\alpha = N_\alpha$. Indeed, let $N_0^\alpha = N_\gamma$, where $\gamma < \Lambda$. Since $N_\alpha = N_\beta^\alpha$ for some β, it follows that $\gamma \leqslant \alpha$. For each $\xi \leqslant \alpha$, let $\varphi(\xi)$ be the ordinal number such that $N_0^\xi = N_{\varphi(\xi)}$, where N_0^0 is defined to be N_0. φ is strictly increasing and, by considering the cases where ξ is a successor or a limit ordinal, it is seen by transfinite induction that $\xi \leqslant \varphi(\xi)$ for all $\xi \leqslant \alpha$. Therefore, $\gamma = \alpha$, and M_α is an $\mathscr{N}$-set contained in N which is disjoint from $X - U$. It follows that $X - U$ is $\mathscr{N}$-singular.

LEMMA 2. *If $\mathscr{C}$ is an $\mathfrak{M}$-family of subsets of X and $\mathscr{N}$ is a subfamily of $\mathscr{C}$ such that*

$$(*) \qquad \text{each } \mathscr{C}\text{-set contains an } \mathscr{N}\text{-set},$$

then $\mathscr{N}$ is an $\mathfrak{M}$-family of subsets of $Y = \bigcup \mathscr{N}$ and the $\mathscr{N}$-singular sets coincide with the $\mathscr{C}$-singular subsets of Y. Also, if U is a subset of Y and $Y - U$ is $\mathscr{N}$-singular, then $X - U$ is $\mathscr{C}$-singular.

Proof. Axioms 1 and 2 are obviously satisfied. As for Axiom 3, if A is an $\mathscr{N}$-set, then $A \cap (\bigcup \mathscr{D})$ contains an $\mathscr{N}$-set if and only if it contains a $\mathscr{C}$-set. Thus, Axiom 3 follows from $(*)$, as does the fact that the $\mathscr{N}$-singular and $\mathscr{C}$-singular subsets of Y coincide. Finally, $X - U = (X - Y) \cup \cup (Y - U)$ is the union of two $\mathscr{C}$-singular sets whenever $Y - U$ is $\mathscr{N}$-singular.

THEOREM 1. *Let $\mathscr{C}$ be an $\mathfrak{M}$-family of subsets of X and let S be a subset of X. Player* II *has a winning strategy in the game $\Gamma(S, \mathscr{C})$ if and only if S is a $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. Assume throughout that $\mathscr{C}$ is well ordered. If S is a $\mathscr{C}_{\mathrm{I}}$-set, then player II, clearly, has a winning strategy.

Conversely, suppose τ is a winning strategy for player II. Following [10], a τ-chain of order n is a descending-finite sequence $\langle A_1, \ldots, A_{2n}\rangle$ of even length $2n$ in $\mathscr{C}$ satisfying $A_{2k} = \tau(A_1, \ldots, A_{2k-1})$ for $1 \leqslant k \leqslant n$. A τ-chain of order $n+k$ is a continuation of a τ-chain of order n if the first $2n$ terms of both chains are the same. If $\{\langle A_1^t, \ldots, A_{2n}^t\rangle \mid t \in T\}$ is a family of τ-chains of order n, then the intersection and union of these τ-chains are the sets $\bigcap_{t \in T} A_{2n}^t$ and $\bigcup_{t \in T} A_{2n}^t$, respectively. If $\mathscr{W} = \langle A_1, \ldots, A_{2n}\rangle$ is a τ-chain of order n, then $x \in \mathscr{W}$ means $x \in A_{2n}$, and $\mathscr{W}$ is said to be *contained in a set B* if $A_{2n} \subseteq B$.

Let $\mathscr{N}_1$ be the family of all τ-chains of order 1. For each $\mathscr{C}$-set A, there is a τ-chain of order 1 contained in A, e.g. $\langle A, \tau(A)\rangle$. Thus, Lemmas 1 and 2 can be applied to $\mathscr{N}_1$ to obtain a maximal subfamily $\mathscr{M}_1$ of disjoint τ-chains of order 1, the complement of whose union is $\mathscr{C}$-singular. Assume, for a given positive integer k, a maximal family $\mathscr{M}_k$ of disjoint τ-chains of order k has been defined by means of Lemmas 1 and 2. Let $\mathscr{N}_{k+1}$ denote the family of all τ-chains of order $k+1$ which are continuations of τ-chains in $\mathscr{M}_k$. It will now be shown that each $\mathscr{C}$-set contains a τ-chain belonging to $\mathscr{N}_{k+1}$. Suppose A is a $\mathscr{C}$-set. By induction, there is a τ-chain $\langle A_1, \ldots, A_{2k}\rangle$ belonging to $\mathscr{N}_k$ such that $A_{2k} \subseteq A$. Let $\mathscr{N} = \mathscr{N}_k$ and $\mathscr{M} = \mathscr{M}_k$ in the proof of Lemma 1 and suppose $A_{2k} = N_\alpha$, where $\alpha < \Lambda$. If $\alpha = 0$, then $M_0 = N_0$, and hence if $\langle B_1, \ldots, B_{2k}\rangle$ is the τ-chain in $\mathscr{M}_k$ with $B_{2k} = M_0$, then $\langle B_1, \ldots, B_{2k}, M_0, \tau(M_0)\rangle$ is a member of $\mathscr{N}_{k+1}$ contained in A. Thus, assume $\alpha > 0$. If $N_\alpha \cap (\bigcup_{\beta<\alpha} M_\beta)$ contains a member of $\mathscr{N}_k$, then there is an ordinal number $\beta < \alpha$ such that $N_\alpha \cap M_\beta$ contains a member B

of $\mathscr{N}_k$. Hence, if $\langle B_1, \ldots, B_{2k}\rangle$ is the τ-chain in $\mathscr{M}_k$ with $B_{2k} = M_\beta$, then $\langle B_1, \ldots, B_{2k}, B, \tau(B)\rangle$ is a member of $\mathscr{N}_{k+1}$ contained in A. On the other hand, if $N_\alpha \cap (\bigcup_{\beta<\alpha} M_\beta)$ contains no member of $\mathscr{N}_k$, then $M_\alpha \subseteq N_\alpha$, so that if $\langle B_1, \ldots, B_{2k}\rangle$ is the τ-chain in $\mathscr{M}_k$ with $B_{2k} = M_\alpha$, then $\langle B_1, \ldots, B_{2k}, M_\alpha, \tau(M_\alpha)\rangle$ is a member of $\mathscr{N}_{k+1}$ contained in A. Therefore, condition (*) holds for $\mathscr{N}_{k+1}$ and the lemmas can be again applied to obtain a maximal subfamily $\mathscr{M}_{k+1}$ of $\mathscr{N}_{k+1}$ consisting of disjoint τ-chains of order $k+1$ such that $X - \bigcup \mathscr{M}_{k+1}$ is $\mathscr{C}$-singular. It then follows, as in [10], that S is a $\mathscr{C}_{\mathrm{I}}$-set.

A consequence of this characterization of the sets S, for which $\Gamma(S, \mathscr{C})$ is determined in favor of player II, is the following generalization of a result of Lebesgue (see [3], p. 185-186, and [10]):

COROLLARY 1. *For any $\mathscr{C}_{\mathrm{II}}$-set S, there is a $\mathscr{C}$-set A such that, for every $\mathscr{C}$-set $B \subseteq A$, $S \cap B$ is a $\mathscr{C}_{\mathrm{II}}$-set.*

Proof. If, for every $\mathscr{C}$-set A, there is a $\mathscr{C}$-set $B \subseteq A$ such that $S \cap B$ is a $\mathscr{C}_{\mathrm{I}}$-set, then player II has a winning strategy in the game $\Gamma(S, \mathscr{C})$ and, consequently, S must be a $\mathscr{C}_{\mathrm{I}}$-set.

Remark. Note that only Axiom 3 was used in the proof of Theorem 1. In the generalization of the Mazur game to a topological space given by Oxtoby, the family $\mathscr{C}$ is assumed to satisfy the following conditions:

(i) every $\mathscr{C}$-set has a non-empty interior, and

(ii) every non-empty open set contains a $\mathscr{C}$-set.

In general, such a family is not an $\mathfrak{M}$-family. However, if $(X, \mathscr{T})$ is a topological space and $\mathscr{C}$ is a subfamily of $\mathscr{T}$ satisfying Oxtoby's conditions, then $\mathscr{C}$ satisfies Axiom 3.

THEOREM 2. *Let $\mathscr{C}$ be an $\mathfrak{M}$-family of subsets of X satisfying the following axiom:*

4. *There is a sequence $\langle h_n\rangle_{n=1}^{\infty}$ of functions mapping $\mathscr{C}$ into $\mathscr{C}$ such that*

a. *for each $\mathscr{C}$-set A, $h_n(A) \subseteq A$; and*

b. *for every sequence $\langle A_n\rangle_{n=1}^{\infty}$ of $\mathscr{C}$-sets, if $\langle h_n(A_n)\rangle_{n=1}^{\infty}$ is a descending sequence, then $\bigcap_{n=1}^{\infty} h_n(A_n)$ contains only one point.*

Then player I *has a winning strategy in the game $\Gamma(S, \mathscr{C})$ if and only if there is a $\mathscr{C}$-set E such that $E \cap (X - S)$ is a $\mathscr{C}_{\mathrm{I}}$-set.*

Proof. If there is such a $\mathscr{C}$-set E, then player I, obviously, has a winning strategy. Conversely, assume σ is a winning strategy for player I and let $\mathscr{C}$ be well ordered. It will be shown that $E \cap (X - S)$, where $E = \sigma(\theta)$, is a $\mathscr{C}_{\mathrm{I}}$-set. Define, as in Oxtoby's paper, a new winning strategy σ^* for player I, with $\sigma^*(\theta) = \sigma(\theta) = E$, such that the result of each play consistent with σ^* is a singleton. Use σ^* to define a winning strategy τ for player II in the game $\Gamma\big(E \cap (X - S), \mathscr{C}\big)$ in the following manner. Let A_1 be a $\mathscr{C}$-set.

If $A_1 \cap E$ contains no $\mathscr{C}$-set, then there is a $\mathscr{C}$-set $B \subseteq A_1 - E$. Let $\tau(A_1)$ be the first such set B and define $\tau(A_1, \ldots, A_{2n+1}) = A_{2n+1}$ for $n \geqslant 1$. On the other hand, if $A_1 \cap E$ contains a $\mathscr{C}$-set, let B_1 be the first such $\mathscr{C}$-set; define $\tau(A_1) = \sigma^*(E, B_1)$ and, for $n \geqslant 1$, define

$$\tau(A_1, \ldots, A_{2n+1}) = \sigma^*(E, B_1, A_2, A_3, \ldots, A_{2n+1}).$$

By Theorem 1, $E \cap (X - S)$ is a $\mathscr{C}_{\mathrm{I}}$-set.

THEOREM 3. *If $\mathscr{C}$ is an $\mathfrak{M}$-family of subsets of X satisfying Axiom 4 and S is a subset of X which has the Baire property with respect to $\mathscr{C}$, then the game $\Gamma(S, \mathscr{C})$ is determined.*

Proof. If player II has no winning strategy, then S is a $\mathscr{C}_{\mathrm{II}}$-set. Hence, there is a $\mathscr{C}$-set A such that, for every $\mathscr{C}$-set $B \subseteq A$, $B \cap S$ is a $\mathscr{C}_{\mathrm{II}}$-set. Since S has the Baire property, there is a $\mathscr{C}$-set $E \subseteq A$ such that $E \cap (X - S)$ is a $\mathscr{C}_{\mathrm{I}}$-set. Therefore, player I has a winning strategy.

4. Examples of $\mathfrak{M}$-families.

Example 1. Let $X \subseteq [0, 1]$ be a perfect set and let $\mathscr{C}$ be the family of all perfect sets of the form $X \cap I$, where I is a closed subinterval of $[0, 1]$. Axioms 1 and 2 are, obviously, satisfied. Axiom 3 follows from the fact that if A and D are $\mathscr{C}$-sets, then $A \cap D$ is either uncountable and, hence, contains a $\mathscr{C}$-set or it contains at most two points. For, if $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then $A \cap D$ contains a $\mathscr{C}$-set for some $\mathscr{D}$-set D. On the other hand, if $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set and $p, q \in A$ are two-sided limit points of A (i.e., every open interval containing p (respectively, q) also contains infinitely many points less than p (respectively, q) and infinitely many points greater than p (respectively, q)), then $X \cap [p, q]$ is a $\mathscr{C}$-set contained in A and disjoint from every $\mathscr{D}$-set. Therefore, $\mathscr{C}$ is an $\mathfrak{M}$-family. Assume now that $\mathscr{C}$ is well-ordered and, for each positive integer n and each $\mathscr{C}$-set A, define $h_n(A)$ to be the first $\mathscr{C}$-set $B \subseteq A$ whose diameter is less than or equal to $1/n$. Axiom 4 is thus seen to hold.

In this example, the $\mathscr{C}$-singular sets coincide with the sets which are nowhere dense relative to X, the $\mathscr{C}_{\mathrm{I}}$-sets coincide with the sets which are of the first category relative to X, and the sets having the Baire property with respect to $\mathscr{C}$ are the sets having the classical Baire property relative to X.

Note that, by virtue of Lemma 2, the family of all closed subintervals of $[0, 1]$ with non-empty interiors and rational end points is an $\mathfrak{M}$-family which determines the same $\mathscr{C}$-singular sets as the $\mathfrak{M}$-family $\mathscr{C} = \{[a, b] \mid 0 \leqslant a < b \leqslant 1\}$.

Example 2. Another important case included both in Oxtoby's framework and the above framework is where $X = E^\infty$ is the Baire space of all infinite sequences of points belonging to a set E and $\mathscr{C}$ consists of all Baire intervals. Axioms 1 and 2 are, obviously, satisfied, Axiom 3 follows from the remark after Corollary 1, and Axiom 4 is satisfied when, for each n, h_n assigns to each Baire interval a Baire subinterval of rank greater than or equal to n. Thus, $\mathscr{C}$ is an $\mathfrak{M}$-family satisfying Axiom 4 and the $\mathscr{C}_{\mathrm{I}}$-sets are precisely the sets of the first category.

Example 3. Let f be a non-constant, monotone increasing, continuous function on $X = [0, 1]$, let μ_f be the Lebesgue-Stieltjes measure induced by f, and let $\mathscr{C}$ be the family of all closed subsets of X which have positive μ_f-measure. Axioms 1 and 2 are satisfied as well as Axiom 3 upon observing that $\mathscr{D}$ must be countable. If functions h_n are defined as in Example 1, then Axiom 4 also holds.

By a result of Burstin [2], a set $S \subseteq X$ is μ_f-measurable if and only if there is no perfect set P of positive μ_f-measure such that every perfect set $Q \subseteq P$ of positive μ_f-measure contains points of S and points of $X - S$. Thus it follows that the $\mathscr{C}$-singular sets are precisely the sets of μ_f-measure zero and the sets which have the Baire property with respect to $\mathscr{C}$ coincide with the μ_f-measurable sets. Note that the same class of singular sets is obtained upon replacing $\mathscr{C}$ by the family of all nowhere dense perfect sets of positive μ_f-measure.

Remark. J. C. Oxtoby has informed me that, in the case of Lebesgue measure, this game is also included in his generalization; e.g., consider $[0, 1]$ with the density topology and let $\mathscr{C}$ be all ordinary closed sets of positive measure. The solution to this game has also been given essentially by Mycielski [8].

Example 4. Let X be the unit interval and let $\mathscr{C}$ be the family of all perfect sets contained in X. Axioms 1 and 2 are, obviously, satisfied as is also Axiom 4 when the functions h_n are defined as in Example 1. As for Axiom 3, if $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set, then, by the Cantor-Bendixson Theorem, $A \cap D$ is countable for each $D \in \mathscr{D}$. The set $A \cap (\bigcup \mathscr{D})$ can, therefore, be expressed as a countable union of sets each of which has power less than the power of the continuum. By a theorem in Lusin and Sierpiński [6], the proof of which may be viewed as the first occasion upon which the present game was played, $A \cap (\bigcup \mathscr{D})$ has power less than the power of the continuum. Since, as also shown by Lusin and Sierpiński, every perfect set in $[0, 1]$ contains continuum many disjoint perfect sets, there is a $\mathscr{C}$-set $B \subseteq A$ which is disjoint from every $\mathscr{D}$-set. If $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then $A \cap D$ must be uncountable for some set $D \in \mathscr{D}$ and, hence, there is a $\mathscr{C}$-set contained in $A \cap D$. Thus, Axiom 3 is satisfied.

In this example, the $\mathscr{C}$-singular sets and the $\mathscr{C}_{\mathrm{I}}$-sets coincide. Included among the $\mathscr{C}$-singular sets are the "singular sets" discovered by Lusin (see [4] and [5]) (originally, under the assumption of the Continuum Hypothesis); e.g., uncountable sets which are of the first category relative to every perfect set. If there exists a set of the second category which has power less than the power of the continuum, then not all $\mathscr{C}$-singular sets are of the first category.

Note that if $\mathscr{N}$ is the family of all nowhere dense perfect sets or the family of all perfect sets of Lebesgue measure zero, then it follows from Lemma 2 that $\mathscr{N}$ is also an $\mathfrak{M}$-family and determines the same family of singular sets as does $\mathscr{C}$.

Example 5. Let X be an ordered set with order type $1+\lambda+1$ (i.e., with the same order type as the unit interval $[0,1]$) and let $\mathscr{C}$ be the family of all subsets of X with order type $1+\lambda+1$. To verify that Axiom 1 is satisfied, a generalization of the classical "Axiom of Ascoli" will be established (see [13], p. 201 and 202).

Proposition 1. *Let X be an ordered set. If $\langle A_n\rangle_{n=0}^{\infty}$ is a descending sequence of subsets of X and each A_n has order type $1+\lambda+1$, then the intersection of the A_n's is non-empty.*

Proof. Assume the conclusion does not hold. For each n, set $a_n = \inf A_n$ and $b_n = \sup A_n$. The sequence $\langle a_n\rangle_{n=0}^{\infty}$ is monotone increasing, the sequence $\langle b_m\rangle_{m=0}^{\infty}$ is monotone decreasing, and $a_n < b_m$ for all n, m. Moreover, each of these sequences has infinitely many distinct terms. For simplicity, assume 0 is a limit ordinal and proceed by transfinite induction on the set of all limit ordinals less than the first uncountable ordinal Ω. Let β be a limit ordinal and assume, for all limit ordinals $\xi < \beta$, a monotone increasing sequence $\langle a_{\xi+n}\rangle_{n=0}^{\infty}$ containing infinitely many distinct terms has been defined such that $a_{\xi+n}\in A_n$ for all n, $a_{\xi+n} < b_m$ for all n, m, and $a_\gamma < a_\xi$ for all $\gamma < \xi$. There is an increasing sequence $\langle \xi_n\rangle_{n=0}^{\infty}$ of ordinal numbers such that

$$\beta = \sup_n \xi_n,$$

$\langle a_{\xi_n}\rangle_{n=0}^{\infty}$ is a monotone increasing sequence with infinitely many distinct terms, $a_{\xi_n}\in A_n$ for each n, and $a_{\xi_n} < b_m$ for all n, m. If there is a largest limit ordinal γ less than β, then the desired sequence can be obtained upon setting $\xi_n = \gamma+n$. Otherwise, there is an increasing sequence $\langle \gamma_n\rangle_{n=0}^{\infty}$ of limit ordinals such that

$$\beta = \sup_n \gamma_n,$$

in which case, set $\xi_n = \gamma_n+n$. Define, for each n,

$$C_n = \{x\in A_n \mid x \leqslant a_{\xi_k} \text{ for some } k\}$$

and

$$D_n = \{x\in A_n \mid x > a_{\xi_k} \text{ for all } k\}.$$

The pair (C_n, D_n) constitutes a cut of A_n and, hence, determines an element $a_{\beta+n}\in D_n$. It follows that $\langle a_{\beta+n}\rangle_{n=0}^{\infty}$ is a monotone increasing sequence in A_0 with infinitely many distinct terms such that $a_{\beta+n} \in A_n$ for all n, $a_{\beta+n} < b_m$ for all n, m, and $a_\gamma < a_\beta$ for all ordinal numbers $\gamma < \beta$. Thus, it is defined a subset $\{a_\beta \mid \beta < \Omega\}$ of A_0 with order type Ω. However, a set of order type $1+\lambda+1$ can have no subset of order type Ω. Therefore, the intersection of the A_n's is non-empty.

Thus, Axiom 1 is satisfied. Axiom 2 is easily verified and Axiom 3 follows in a manner analogous to that used in Example 4 by virtue of the fact that every ordered set of order type $1+\lambda+1$ contains continuum many disjoint sets of order type $1+\lambda+1$ (see [13], p. 219 and 220) and the fact that if A and D are $\mathscr{C}$-sets and $A \cap D$ is uncountable, then $A \cap D$ contains a $\mathscr{C}$-set. To prove the latter result, let X be mapped order-isomorphically onto the unit interval. Now, any subset of $[0, 1]$ having order type $1+\lambda+1$ is an uncountable G_δ-set (see [13], p. 219). Thus, if A and D are two subsets of $[0, 1]$ which have order type $1+\lambda+1$ and $A \cap D$ is uncountable, then $A \cap D$ is an uncountable G_δ-set. By a result of Young [14], $A \cap D$ contains a perfect set and, hence, a nowhere dense perfect set. Finally, any bounded, nowhere dense perfect set is order isomorphic to the Cantor set, and upon removing from the Cantor set the "left end-points"

$$\frac{1}{3}, \frac{1}{9}, \frac{7}{9}, \frac{1}{27}, \frac{7}{27}, \frac{19}{27}, \frac{25}{27}, \ldots$$

in $(0, 1)$, a set of order type $1+\lambda+1$ is obtained.

To verify Axiom 4, let $a_1, a_2, \ldots$ be a countable order-dense subset of X and let $\mathscr{C}$ be well ordered. For each n and each $\mathscr{C}$-set A, let $h_n(A)$ be the first $\mathscr{C}$-set $B \subseteq A$ such that either $x < a_n$ for all $x \in B$ or $x > a_n$ for all $x \in B$. Since between any two points of X there is a point a_k, if $\langle A_n \rangle_{n=1}^{\infty}$ is any sequence in $\mathscr{C}$ and $\langle h_n(A_n) \rangle_{n=1}^{\infty}$ is a descending sequence, then $\bigcap_{n=1}^{\infty} h_n(A_n)$ contains only one point.

Remark. Consider the special case where $X = [0, 1]$. Due to the fact that every set of order type $1+\lambda+1$ contains a perfect set, and, conversely, the $\mathscr{C}$-singular sets in this example and Example 4 coincide.

In the next two examples, the terminology and theorems of [12] will be used.

Example 6. Assume the Continuum Hypothesis. Let $X = [0, 1]$, let h be a continuous function in $\mathscr{H}_0$, let μ^h be the Hausdorff measure associated with h, and let $\mathscr{C}$ be the family of all closed sets with positive μ^h-measure. Axioms 1, 2a, and 4, obviously, hold. For Axiom 2b, take an ascending sequence $\langle A_n \rangle_{n=1}^{\infty}$ of closed sets such that

$$\bigcup_{n=1}^{\infty} A_n = A - \{x\}$$

and, then, choose an n for which $\mu^h(A_n) > 0$. As for Axiom 3, if $A \cap (\bigcup \mathscr{D})$ contains a $\mathscr{C}$-set, then $A \cap D$ is a $\mathscr{C}$-set for some $D \in \mathscr{D}$. If $A \cap (\bigcup \mathscr{D})$ contains no $\mathscr{C}$-set, then $A \cap D$ has μ^h-measure zero for all $D \in \mathscr{D}$, so that $A - (A \cap (\bigcup \mathscr{D}))$ is a G_δ-set of positive μ^h-measure and, consequently, contains a $\mathscr{C}$-set. Thus Axiom 3 holds. In this example, the $\mathscr{C}$-singular sets include all Borel sets of μ^h-measure zero.

Example 7. Assume the Continuum Hypothesis. Let $X = [0, 1]$ and let $\mathscr{C}$ be the family of all closed sets of Hausdorff dimension a for some a with $0 < a \leqslant 1$; then Axioms 1-4 are satisfied. In this example, all Borel sets of dimension zero are $\mathscr{C}$-singular sets.

Example 8. Let X be an uncountable set, let $\mathscr{I}$ be a proper σ-ideal of subsets of X containing all singletons, and let $\mathscr{C} = \{A \mid X - A \in \mathscr{I}\}$. $\mathscr{C}$ is an $\mathfrak{M}$-family, the class of $\mathscr{C}$-singular sets coincides with $\mathscr{I}$, and the family of all sets which have the Baire property with respect to $\mathscr{C}$ is just $\mathscr{C} \cup \mathscr{I}$. Note that $\mathscr{C}$ does not satisfy Axiom 4.

Remark. Some very interesting new ideals of sets on the real line which have arisen from game-theoretical considerations have recently been investigated by Mycielski [9].

5. Problems. The first two problems are in the terminology of [1].

PROBLEM 1. Does the family of all perfect M-sets form an $\mathfrak{M}$-family? (**P 870**)

Remark. Assuming the Continuum Hypothesis it can be shown that the family of all closed sets which support a measure μ, not identically zero, whose Fourier-Stieltjes transform $\hat{\mu}(t) \to 0$ as $|t| \to \infty$, is an $\mathfrak{M}$-family.

PROBLEM 2. Does the family of all perfect sets which are not R-sets form an $\mathfrak{M}$-family? (**P 871**)

PROBLEM 3. Can the Continuum Hypothesis be dropped in Examples 6 and 7? (**P 872**)

PROBLEM 4. Does the family of sets which have the Baire property in the sense of Example 6 coincide with the family of μ^h-measurable sets? (**P 873**)

6. Concluding remarks. The concept of an $\mathfrak{M}$-family has led the author to an abstract theory of Baire category which will appear shortly (see [7]). Most of the known analogies between Baire category and Lebesgue measure can be subsumed under this general theory (see the book [11] of Oxtoby for a full discussion of these analogies).

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