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VARIETIES OF TOPOLOGICAL GROUPS A SURVEY*

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1. Varieties of groups. The study of varieties of groups (more generally, varieties of algebras) has its roots in the work of Birkhoff [4] and Neumann [60] in the 1930's. A *variety of groups* [61] is the class of all groups satisfying a certain family of laws; for example, the class of all abelian groups of exponent (dividing) n satisfies the laws $x^{-1}y^{-1}xy = 1$ and $x^n = 1$. From this definition it is immediately clear that there are no more than $2^{\aleph_0}$ varieties of groups. But it was not until 1970 that Ol'šanskiĭ [63] showed that there are precisely $2^{\aleph_0}$ varieties of groups. We will see that this is very different from the situation for varieties of topological groups.

Birkhoff observed that there is an alternative, but equivalent, definition for varieties of groups. A variety of groups can be described as a class of groups closed under the operations Q, S, and C, where Q denotes a quotient group, S a subgroup, and C an arbitrary Cartesian product. (To be pedantic one should say that varieties are also closed under the formation of isomorphic images.) Indeed, Birkhoff proved the following

THEOREM 1 ([61]). *If Ω is any non-empty class of groups and $V(\Omega)$ is the smallest variety containing Ω, then $V(\Omega) = QSC(\Omega)$.*

So every member of $V(\Omega)$ can be written as a quotient group of a subgroup of some Cartesian product of members of Ω.

We could now turn to a discussion of varieties of topological groups, but instead we next look at varieties of topological vector spaces, as certain features of topological varieties present themselves more clearly.

2. Varieties of topological vector spaces. A non-empty class of locally convex Hausdorff topological vector spaces (LCS's) is said to be a *variety* [13], [14] if it is closed under the operations Q, S, and C, where Q denotes a quotient space, S a (not necessarily closed) subspace, and C an arbitrary Cartesian product with the Tychonoff product topology. Some well-

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known classes of LCS's form varieties; for example, the class of all Schwartz spaces, nuclear spaces, and LCS's having the weak topology.

In due course we will discuss topological laws and see how topological varieties can be described in terms of laws.

To see how far topological varieties are from algebraic varieties, we introduce the concept of a $T(m)$-space.

If E is an LCS and m is an infinite cardinal, we say that E is a $T(m)$-*space* if every neighbourhood of 0 in E contains a subspace of E of codimension strictly less than m. As every neighbourhood of 0 contains the trivial subspace $\{0\}$, it is clear that an LCS F is a $T(m)$-space for every m strictly greater than the dimension of F. Of course, the interest lies in how small an m will suffice. Any normed space N is a $T(m)$-space if and only if the dimension of N is strictly less than m. (To see this, simply look at the unit ball in N.) However, all LCS's with the weak topology (and these spaces can have arbitrarily large dimension) are $T(\aleph_0)$-spaces. (Indeed, this is a characterization of such spaces.)

As products, subspaces, and quotients of $T(m)$-spaces are $T(m)$-spaces, we obtain

THEOREM 2 ([14]). *If m is any infinite cardinal number and Ω is a class of $T(m)$-spaces, then $V(\Omega)$ contains only $T(m)$-spaces.*

COROLLARY 1. *The class of all $T(m)$-spaces for any infinite cardinal number m is a variety.*

COROLLARY 2. *If N is an infinite-dimensional normed space and E is an LCS of smaller dimension, then $N \notin V(E)$.*

As there are normed spaces of arbitrarily large dimension, we are led to the following unexpected result:

COROLLARY 3. *There is a proper class of varieties of topological vector spaces.*

It is easily verified that every $T(m)$-space E is isomorphic to a subspace of a product $\prod_{i \in I} E_i$, where each E_i is a quotient space of E and has dimension strictly less than m. This observation has some important consequences.

COROLLARY 4. *Every variety of $T(m)$-spaces is generated by LCS's of dimension strictly less than m.*

Up to isomorphism there is only a *set* of LCS's of dimension less than m. So any variety generated by a family of LCS's, each of dimension less than m, is also *singly generated*; i.e. generated by a single LCS (namely, the product of the members of the family).

COROLLARY 5. *Every variety of $T(m)$-spaces is singly generated.*

COROLLARY 6. *Every subvariety of a singly generated variety is singly generated.*

We now state the most powerful known result in the study of varieties of LCS's. A proof of the analogous result for varieties of topological groups will be given later.

THEOREM 3 ([14]). *Let E be an LCS. If Ω is any non-empty class of LCS's, then $E \in V(\Omega)$ if and only if $E \in SCQP(\Omega)$, where P denotes a finite product.*

An LCS E is said to be a *universal generator* for a variety V if $V = SC(\{E\})$. A deep result of Kōmura and Kōmura [30] states that the space (s) of rapidly decreasing sequence is a universal generator for the variety of nuclear spaces. We now show how variety theory implies that the nuclear variety, the Schwartz variety, and others have universal generators. As an immediate consequence of Theorem 3 we have

COROLLARY 1. *Every singly generated variety has a universal generator.*

Using this together with Corollary 6 to Theorem 2 and the fact that the nuclear variety is a subvariety of the Schwartz variety which is in turn a subvariety of $V(l_1)$ we obtain

COROLLARY 2. *The variety of nuclear spaces and the variety of Schwartz spaces have universal generators.*

Of course, Corollary 2 does not give us a concrete universal generator. As stated earlier, Kōmura and Kōmura [30] have such a generator for the nuclear variety, while Randtke [66] and Jarchow [27] have found universal generators for the Schwartz variety.

We now state a simple lemma which shows why Theorem 3 is important in the study of Banach spaces in varieties.

LEMMA. *Let N be a normed space. Then if $N \in SC(\Omega)$ for some class Ω of LCS's, then $N \in SP(\Omega)$.*

Using Theorem 3 and this lemma we obtain

THEOREM 4. *Let Ω be a class of LCS's and N a normed space in $V(\Omega)$ Then $N \in SPQP(\Omega) = SQP(\Omega)$.*

COROLLARY 1. *Let Ω be a class of Banach spaces each of which has property $\mathscr{P}$, where $\mathscr{P}$ is one of the following: reflexive, quasi-reflexive* [11], *almost reflexive* [31], *separable, having separable dual, Hilbert. Then every Banach space in $V(\Omega)$ also has property $\mathscr{P}$.*

Theorem 4 also allows us to use Banach space techniques to analyze the varieties generated by the classical Banach spaces. This is done in [14] and [56]. We record here some interesting results.

COROLLARY 2 ([14]). *For $1 < p < \infty$ and $p \neq 2$, K any uncountable compact metric space, (s) the space of rapidly decreasing sequences* [30], *and φ the strong dual of the Fréchet space of all real sequences with the product* *topology* [14],

$$V(R) \subsetneqq V(\varphi) \subsetneqq V\big((s)\big) \subsetneqq V(l_p) \subsetneqq V(L_p)$$
$$\subsetneqq V(l_1) = V\big(C(K)\big) = V(L_1) \subsetneqq V(l_\infty) = V(L_\infty).$$

The next corollary is a much improved version of Banach's result on the incomparable linear dimension of l_p and l_q for $p \neq q$.

COROLLARY 3 ([56]). *If* $1 < p, q < \infty$ *and* $p \neq q$, *then* $V(l_p) \cap V(l_q)$ *is contained in the variety of Schwartz spaces. So any normed space in* $V(l_p) \cap V(l_q)$ *is finite dimensional.*

COROLLARY 4 ([56]). *If the infinite-dimensional Banach space* B *is reflexive, quasi-reflexive or has a separable second dual, then* $V(B) \cap V(c_0)$ *is a subvariety of the variety of Schwartz spaces.*

Very little, however, is known about varieties generated by two or more classical Banach spaces. For example, we record the open question:

PROBLEM. If $1 < p, q < \infty$ and $p \neq q$, for what r does l_r belong to $V(\{l_p, l_q\})$? (**P 1248**)

For $1 < p < \infty$ and $p \neq 2$, $l_r \in V(\{l_p, l_2\})$ if and only if r is between p and 2 (see [56]).

Many other questions about varieties of LCS's are mentioned in [14] and [56].

3. Varieties of topological groups. In 1952 Higman [23] suggested one candidate for the definition of a variety of topological groups, however his work has not been followed up. In 1968 Ian D. Macdonald suggested to the author that a *variety of topological groups* should be defined to be a class of (not necessarily Hausdorff) topological groups closed under the operations Q, S, and C, where Q denotes a (not necessarily Hausdorff) quotient group, S a subgroup with the relative topology, and C an arbitrary Cartesian product with the Tychonoff topology. By not restricting to Hausdorff groups, the underlying class of groups forms a variety of groups. Two interesting types of varieties are the wide varieties and the full varieties. A variety of topological groups is said to be a *wide variety* if it is closed under the operation H, where H denotes a continuous homomorphic image. If V is a variety of groups, the *full variety* corresponding to V is the class of all topological groups with underlying group lying in V.

As for varieties of LCS's the $T(m)$-property plays an important role. A topological group G is said to be an $S(m)$-*group* if it has a basis of open neighbourhoods of the identity consisting of subgroups of G of index strictly less than m, where m is an infinite cardinal number. A $T(m)$-*group* is defined to be a continuous homomorphic image of an $S(m)$-group. Of course, every $S(m)$-group is a $T(m)$-group while the converse is patently false. Every topological group G is a $T(m)$-group for every m greater than the cardinality of G. However, a Lie group (in particular, a discrete group) G is a $T(m)$-group if and only if m is strictly greater than the cardinality of G. (This follows from the fact that Lie groups have no small subgroups.) However, every compact Hausdorff group, being topologically isomorphic to a subgroup of a product of unitary groups, is a $T(2^{2^{\aleph_0}})$-group.

THEOREM 5 ([39]). *If Ω is a class of $S(m)$-groups or $T(m)$-groups, then $V(\Omega)$ contains only $S(m)$-groups or $T(m)$-groups, respectively.*

COROLLARY 1. *For every (non-trivial) variety of groups V, there is a proper class of varieties of topological groups having V as their underlying variety of groups.*

COROLLARY 2. *No (non-trivial) full variety of topological groups is singly generated.*

To put Corollary 2 in context we remind the reader that every variety of groups is singly generated.

COROLLARY 3. *If R_d and R are the additive groups of reals with the discrete topology and the usual topology, respectively, then $R \notin V(R_d)$.*

Corollary 3 follows from Theorem 5 and the fact that R_d is an $S(2^{\aleph_0})$-group while R is not. So $V(R_d)$ provides an example of a variety which is not a wide variety. (Perhaps we should also mention that $R_d \notin V(R)$. Indeed, $V(R) \cap V(R_d) = V(Z)$, and so consists of $S(\aleph_1)$-groups, where Z is the discrete group of integers.)

Observing that for any class Ω of topological groups we have $SS(\Omega) = S(\Omega)$, $CC(\Omega) = C(\Omega)$, $QQ(\Omega) = Q(\Omega)$, $CS(\Omega) \subseteq SC(\Omega)$, $CQ(\Omega) \subseteq QC(\Omega)$, and $SQ(\Omega) \subseteq QS(\Omega)$, we obtain the analogue of Birkhoff's Theorem 1.

THEOREM 6 ([5]). *For any non-empty class Ω of topological groups,*

$$V(\Omega) = QSC(\Omega).$$

(It should be noted that the alternative proof of Birkhoff's theorem given in [61], which depends on proving that a certain varietal free group lies in $SC(\Omega)$, does not extend to the topological case. See the Example in [5].)

Theorem 6 is not particularly useful for topological varieties except that, together with the following simple lemma, it allows us to prove the fundamental theorem on generating varieties — Theorem 7.

LEMMA. *Let G be a topological group with subgroups A and B such that B is a closed normal subgroup of A. Let $\bar{A}$ and $\bar{B}$ be the closures in G of A and B, respectively. Then the quotient group A/B is topologically isomorphic to a subgroup of $\bar{A}/\bar{B}$.*

THEOREM 7 ([5]). *If Ω is any class of topological groups and G is a Hausdorff group in $V(\Omega)$, then $G \in SC\bar{Q}\bar{S}P(\Omega)$, where $\bar{Q}$ denotes a Hausdorff quotient group and $\bar{S}$ a closed subgroup.*

Proof. By the Lemma, $\bar{Q}SP(\Omega) \subseteq S\bar{Q}\bar{S}P(\Omega)$. Observing that any Hausdorff group in $SCQ(\Omega)$ must lie in $SC\bar{Q}(\Omega)$, we infer that it suffices to prove that $G \in SCQSP(\Omega)$. By Theorem 1, $G \in \bar{Q}SC(\Omega)$. Therefore, there exist $K_i \in \Omega$, $i \in I$, such that $K = \prod_{i\in I} K_i$ has subgroups A and B, where B is a closed normal subgroup of A and G is topologically isomorphic to the quotient group A/B. Let Σ be the family of all finite subsets of I. For $\sigma \in \Sigma$ put

$$K_\sigma = \prod_{i\in\sigma} K_i.$$

Let p_σ be the projection mapping of K onto K_σ, and h_σ the canonical mapping of $p_\sigma(A)$ onto the quotient group $p_\sigma(A)/p_\sigma(B)$. Define a map $f\colon A \to \prod_{\sigma\in\Sigma} h_\sigma p_\sigma(A)$ by

$$f(a) = \prod_{\sigma\in\Sigma} h_\sigma p_\sigma(a) \quad \text{for all } a \in A;$$

i.e. f is the product of $h_\sigma p_\sigma$, $\sigma \in \Sigma$, and hence is a continuous homomorphism. For some $\tau \in \Sigma$, let

$$O = \prod_{i\in I} O_i,$$

where each O_i is a neighbourhood of the identity in K_i and $O_i = K_i$ for $i \notin \tau$. We prove that f is an open mapping of A onto $f(A)$ by verifying that

$$f(O \cap A) \supseteq \Big(\prod_{\sigma\in\Sigma} D_\sigma\Big) \cap f(A),$$

where $D_\sigma = h_\sigma p_\sigma(A)$ for $\sigma \neq \tau$ and $D_\tau = h_\tau\big(p_\tau(O) \cap p_\tau(A)\big)$. If

$$x \in \Big(\prod_{\sigma\in\Sigma} D_\sigma\Big) \cap f(A),$$

then there exists $a \in A$ such that

$$x = f(a) \in \prod_{\sigma\in\Sigma} D_\sigma,$$

and so there exists $c \in O$ such that $h_\tau p_\tau(c) = h_\tau p_\tau(a)$; i.e. $p_\tau(B)p_\tau(c) = p_\tau(B)p_\tau(a)$. Consequently, there exists $b \in B$ such that $p_\tau(c) = p_\tau(b)p_\tau(a) = p_\tau(ba)$. But this means $p_\tau(ba) \in p_\tau(O)$, and therefore $ba \in O$. This shows that $x \in f(O \cap A)$, since clearly B is in the kernel of f, and so $x = f(a) = f(ba)$.

We now show that the kernel of f is in B. If $y \notin B$, then, since B is closed in A, there exists a neighbourhood $N = \prod_{i\in I} N_i$ of y in $\prod_{i\in I} K_i$ with

$N \cap B = \emptyset$, and $N_i = K$ for all $i \notin \gamma$, where γ is suitably chosen from Σ. Clearly, $p_\gamma(N) \cap p_\gamma(B) = \emptyset$ and since $p_\gamma(y) \in p_\gamma(N)$, we have $p_\gamma(y) \notin p_\gamma(B)$. Since $p_\sigma(B)$ is the kernel of h_σ, this shows that $f(y) = \big(h_\sigma p_\sigma(y)\big)_{\sigma \in \Sigma}$ is not the identity.

Thus f is an open continuous homomorphism with kernel B, and so A/B is topologically isomorphic to a subgroup of $\prod\limits_{\sigma \in \Sigma} h_\sigma p_\sigma(A)$; i.e. $G \in SCQSP(\Omega)$, as we set out to prove.

COROLLARY 1. *If Ω is any class of compact (or even pseudocompact) Hausdorff groups, then every complete Hausdorff (in particular, locally compact Hausdorff) group in $V(\Omega)$ is compact.*

COROLLARY 2. *If Ω is any class of locally compact Hausdorff groups or a class of complete metrizable groups, then every Hausdorff group G in $V(\Omega)$ has a completion $\hat{G}$ which is in $V(\Omega)$.*

In the abelian case the operators Q and S commute, and so Theorem 7 yields

COROLLARY 3. *If Ω is any class of abelian topological groups and G is a Hausdorff group in $V(\Omega)$, then $G \in SC\bar{Q}P(\Omega)$.*

PROBLEM. Does Corollary 3 remain true if the condition "abelian" is removed? (**P 1249**)

Naturally, we are interested in varieties generated by locally compact groups. As a first step we look at varieties generated by Lie groups. One might expect there to be a correspondence between varieties of topological groups generated by Lie groups and varieties of Lie algebras. But it soon becomes evident that one should be considering topological Lie algebras.

A *Banach Lie algebra* L is a Lie algebra (over the reals) and a Banach space such that there is a constant C with $\|[xy]\| \leqslant C\|x\|\,\|y\|$ for each $x, y \in L$. If B_i, $i \in I$, are Banach Lie algebras and E is a subalgebra of the product $\prod\limits_{i \in I} B_i$, then E with the topology induced from the product is said to be a *(locally convex) topological Lie algebra*. A non-empty class of topological Lie algebras is said to be a *variety of topological Lie algebras* if it is closed under the formation of Cartesian products, subalgebras, and separated quotients. Note that every variety of topological vector spaces is a variety of topological Lie algebras.

The basic theorem connecting varieties of topological groups generated by Lie groups and varieties of topological Lie algebras is the following

THEOREM 8 ([8]). *Let G be a Lie group and $L(G)$ its topological Lie algebra (i.e., the Lie algebra of G given the unique vector space topology it admits). Let $V(G)$ be the variety of topological groups generated by G, and $V\big(L(G)\big)$ the variety of topological Lie algebras generated by $L(G)$.*

(i) *If H is a Lie group in $V(G)$, then its topological Lie algebra $L(H)$ is in $V(L(G))$.*

(ii) *If L is any finite-dimensional topological Lie algebra in $V(L(G))$, then there is a Lie group H in $V(G)$ such that $L(H)$ is isomorphic to L.*

Theorem 4 states that if we are looking for a normed space N in a variety of topological vector spaces generated by a class Ω of normed spaces, then it must lie in $QSP(\Omega)$; i.e., N can be "manufactured" from Ω without going outside the class of normed spaces. The importance of Theorem 8 lies in the analogous result holding for Banach Lie algebras, and in particular finite-dimensional Lie algebras.

THEOREM 9 ([8]). *If Ω is any class of topological Lie algebras and B is a Banach Lie algebra in $V(\Omega)$, then $B \in \bar{Q}\bar{S}P(\Omega)$. In particular, this is the case for finite-dimensional topological Lie algebras.*

COROLLARY 1. *Let G be a simple Lie group of dimension n and let $\{G_i\colon i \in I\}$ be Lie groups of dimension less than n. Then $G \notin V(\{G_i\colon i \in I\})$.*

COROLLARY 2. *Let Ω be a class of connected soluble (respectively, nilpotent) Lie groups. Then any connected Lie group in $V(\Omega)$ is soluble (respectively, nilpotent).*

Specific simple Lie groups are examined in [8] where it is shown, for example, that for the unitary groups $U(n)$ and the symplectic groups $\mathrm{Sp}(n)$ the following holds: $U(m) \in V(U(n))$ and $\mathrm{Sp}(m) \in V(\mathrm{Sp}(n))$ if and only if $m \leqslant n$.

In view of Theorems 8 and 9 it is possible that in a variety of topological groups generated by a class Ω of Lie groups every Lie group in $V(\Omega)$ can be "manufactured" from Ω without going outside the class of Lie groups. So we state our next open question:

PROBLEM. If Ω is a class of Lie groups and G is a Lie group in $V(\Omega)$, is it true that $G \in \bar{Q}\bar{S}P(\Omega)$? (**P 1250**)

This is known to be true if G is a discrete group [46]. Other partial results on this problem are given in [50]. In particular, we state

THEOREM 10 ([50]). *If Ω is a class of simply connected soluble Lie groups and G is a Lie group in $V(\Omega)$, then $G \in \bar{Q}\bar{S}P(\Omega)$.*

THEOREM 11 ([50]). *Let Ω be a class of locally compact Hausdorff groups. Then*

(i) *$R \in V(\Omega)$ if and only if $R \in S(\Omega)$;*

(ii) *$Z \in V(\Omega)$ if and only if $Z \in S(\Omega)$;*

(iii) *$T \in V(\Omega)$ if and only if $T \in QS(\Omega)$, where T is the circle group;*

(iv) *if every member of Ω is abelian, then $T \in V(\Omega)$ if and only if $T \in Q(\Omega)$.*

COROLLARY. (i) $V(R) \supsetneqq V(\{T, Z\}) \supsetneqq V(T)$. (ii) $V(\{T, Z\}) \supsetneqq V(Z)$.

One final result in this direction is given in [50]. We say that a topological group G has *property S* if for any class Ω of Lie groups such that $G \in V(\Omega)$ we have $G \in S(\Omega)$.

THEOREM 12 ([50]). *Let G be a compact connected Hausdorff group. Then G has property S if and only if G is a simple simply connected Lie group such that the intersection of all the proper non-trivial subgroups of $Z(G)$ is not $\{1\}$; i.e. those having the following Lie algebras:* (i) A_n, $n+1$ *a prime power*; (ii) B_n, $n \geqslant 2$; (iii) C_n, $n \geqslant 3$; (iv) D_n, $n \geqslant 4$, *and n an odd prime power*; (v) G_2; (vi) F_4; (vii) E_6; (viii) E_7; *and* (ix) E_8.

In [10] Chen and Yoh have generalized the definition of Lie groups so that the class of generalized Lie groups is closed under the operation of Cartesian product.

As is well known, connected locally compact Hausdorff groups can be approximated by (finite-dimensional) Lie groups. This allows us to extend Corollary 2 to Theorem 9 to connected locally compact groups. Indeed, we obtain

THEOREM 13 ([46]). *Let Ω be a class of locally compact Hausdorff groups each of which has the component of the identity soluble (respectively, nilpotent). Then any connected locally compact Hausdorff group in $V(\Omega)$ is soluble (respectively, nilpotent).*

Without the connectedness condition Theorem 13 would be false. However, for Lie groups we have

THEOREM 14 ([46]). *If Ω is a class of soluble locally compact Hausdorff groups, then any Lie group in $V(\Omega)$ is soluble.*

In recent years a great deal of work has been done on compactness conditions in topological groups, see, e.g., [21]. It is of interest to see how these conditions behave with respect to the varietal operations. A topological group is said to be *maximally almost periodic* (*MAP*) if it admits a continuous one-one homomorphism into a compact Hausdorff group. A topological group is said to be an *SIN-group* if every neighbourhood of the identity contains a neighbourhood of the identity invariant under inner automorphisms. The class of MAP groups and the class of SIN-groups have precisely the same intersection with the class of connected locally compact Hausdorff groups – namely those groups of the form $R^n \times K$, where K is compact and n is a non-negative integer.

THEOREM 15 ([47]). *If Ω is a class of SIN-groups, then every member of $V(\Omega)$ is an SIN-group.*

If "SIN" were replaced by "MAP" in Theorem 15, the statement would be false, even if Ω consisted of locally compact groups. However, we have

THEOREM 16 ([47]). *If Ω is a class of locally compact MAP-groups, then every connected Hausdorff group in $V(\Omega)$ is MAP.*

A Hausdorff topological group is said to be an *IN-group* if there is a compact neighbourhood of the identity which is invariant under inner automorphisms. Any locally compact Hausdorff group having the closure of its commutator subgroup compact is an IN-group, and conversely — any connected IN-group has the closure of its commutator subgroup compact.

THEOREM 17 ([57]). *Let Ω be a class of topological groups each of which has the property that the closure of its commutator subgroup is compact. Then every complete Hausdorff group in $V(\Omega)$ has this property.*

COROLLARY 1. *Let Ω be a class of IN-groups. Then any connected locally compact Hausdorff group in $V(\Omega)$ is an IN-group.*

COROLLARY 2. *Let Ω be a class of connected IN-groups. Then any locally compact Hausdorff group in $V(\Omega)$ is an IN-group.*

PROBLEM. Is every locally compact Hausdorff group in a variety generated by IN-groups necessarily an IN-group? **(P 1251)** The varieties generated by R and T are of particular interest.

THEOREM 18 ([43], [24], [6]). *Let Ω be a class of locally compact Hausdorff abelian groups none of which is not totally disconnected. Then*

$$V(\Omega) = \begin{cases} V(T) & \textit{if all members of } \Omega \textit{ are compact,} \\ V(R) & \textit{otherwise.} \end{cases}$$

Further, any connected complete group or complete metrizable group in $V(R)$ is topologically isomorphic to a product of copies of R and Z, and a compact group.

PROBLEM. What varieties lie between $V(T)$ and $V(R)$? Indeed, what can be said about the lattice of subvarieties of $V(R)$? **(P 1252)**

Finally in this section, we recall we said that we considered non-Hausdorff topological groups so that the underlying class of groups of a variety of topological groups is a variety of groups. The following theorem shows that restricting to Hausdorff groups would lose this desirable feature.

THEOREM 19 ([51]). *Let Ω be a class of connected compact groups and let Σ be the class of all groups which, with some Hausdorff topology, appear in $V(\Omega)$. Then Σ is a variety of groups if and only if each member of Ω is abelian.*

In contrast with Theorem 19 we present

THEOREM 20 ([52]). *If G is any connected MAP-group (in particular, compact Hausdorff or locally compact Hausdorff abelian), then the underlying variety of groups of $V(G)$ is either the variety of all groups or the variety of all abelian groups.*

The last theorem in this section generalizes a result of Ian D. Macdonald (private communication) which states that if the group G with some topology lies in a variety V, then G with the indiscrete topology (i.e., the open sets are G and $\emptyset$) also lies in V.

THEOREM 21 ([39]). *Let V be a variety of topological groups and G a topological group. If G/N lies in V, where N is the closure of the identity in G, and the underlying group of G with some topology is in V, then G is in V.*

4. Varietal free topological groups. If V is a variety of topological groups, X is a topological space, and F is a member of V, then F is said to be a *free topological group* of V on X, denoted by $F(X, V)$, if it has the following properties:

(a) X is a subspace of F;

(b) X generates F algebraically;

(c) for any continuous mapping γ of X into any member H of V, there exists a continuous homomorphism Γ of F into H such that $\Gamma|X = \gamma$.

THEOREM 22 ([37]). *Let X be a topological space and V a variety of topological groups. Then*

(i) *$F(X, V)$ is unique (up to isomorphism) if it exists;*

(ii) *$F(X, V)$ exists if any only if there is a member of V which has X as a subspace;*

(iii) *$F(X, V)$ is algebraically the free group on the set X of the underlying variety of groups of V.*

As every topological group is a completely regular space and every subspace of a completely regular space is completely regular, it is clear that $F(X, V)$ cannot exist unless X is completely regular. Conversely, we have

COROLLARY 1 ([37]). *If the variety V contains a topological group with non-trivial path component, then $F(X, V)$ exists for any completely regular space X.*

COROLLARY 2 ([37]). *If the topological group G is in the variety V, then $F(G, V)$ exists. Further, G is a quotient group of $F(G, V)$.*

A variety of topological groups V is said to be a *β-variety* if $F(X, V)$ exists and is Hausdorff for every completely regular Hausdorff space X.

THEOREM 23 ([38], [49]). *Every full variety is a β-variety.*

The converse of Theorem 23 is of course false. Indeed, β-varieties, which are not full varieties, exist in profusion.

THEOREM 24 ([39]). *Let V be any variety of groups. Then there is a proper class of varieties of topological groups which are β-varieties and have V as their underlying variety of groups.*

THEOREM 25 ([49]). *If Ω is any (non-trivial) class of arcwise connected Hausdorff groups, then $V(\Omega)$ is a β-variety.*

As every connected locally compact Hausdorff group can be approximated by Lie groups, we obtain the following pleasant corollary:

COROLLARY. *If Ω is any class of connected locally compact Hausdorff groups, then $V(\Omega)$ is a β-variety.*

We mention in passing that the variety of topological groups generated by the class of all locally compact groups is not as large as one might expect. We record three results in this direction.

THEOREM 26 ([44]). *Let Ω be the class of all locally compact Hausdorff groups. Then the locally convex Hausdorff topological vector space E, regarded as a topological group, is in $V(\Omega)$ if and only if E has its weak topology. In particular, no Banach space, regarded as a topological group, is in $V(\Omega)$.*

If V is the variety of all topological groups, $F(X, V)$, where X is the closed unit interval with its usual topology, will be denoted by $F[0,1]$.

THEOREM 27 ([54]). *If V is any variety of topological groups containing $F[0,1]$, then V contains every locally compact Hausdorff group G which has its quotient group $G/C(G)$ compact, where $C(G)$ is the component of the identity in G. In particular, V contains all connected locally compact Hausdorff groups and all compact Hausdorff groups.*

THEOREM 28 ([54]). *The variety of topological groups generated by the class of all locally compact groups does not contain $F[0,1]$.*

It was shown by Kakutani [28], Nakayama [59], and Gelbaum [19] that if V is the variety of all topological groups or of all abelian topological groups and X is any completely regular Hausdorff space, then $F(X, V)$ is an MAP-group. The next theorem generalizes this.

THEOREM 29 ([52]). *For any non-abelian (respectively, abelian) variety V, $F(X, V)$ exists and is MAP for each completely regular Hausdorff space X if and only if V contains a non-abelian (respectively, abelian) arcwise connected MAP-group.*

COROLLARY. *If V is any variety containing a connected compact Hausdorff non-abelian group, then $F(X, V)$ is MAP for each completely regular Hausdorff space X.*

The next result shows how strong an algebraic restriction $F(X, V)$ being MAP is.

THEOREM 30 ([52]). *Let V be a variety such that $F(X, V)$ is MAP for some non-totally disconnected space X. Then the underlying variety of groups of V is either the class of all groups or all abelian groups.*

Relatively little is known about the topology of $F(X, V)$ in general. However, some information is known when V is a full variety.

A Hausdorff topological space X is said to be a *k_ω-space* if

$$X = \bigcup_{n=1}^{\infty} X_n,$$

where each X_n is compact, and it has the property that a subset A of X

is closed if and only if each $A \cap X_n$ is compact. As examples of k_ω-spaces we mention countable CW-complexes, locally compact σ-compact Hausdorff spaces, and connected locally compact Hausdorff groups, e.g., $R = \bigcup_{n=1}^{\infty} X_n$ is a k_ω-decomposition of R if X_n is the interval $[-n, n]$.

THEOREM 31 ([32]). *If X is any k_ω-space and V is a full variety, then $F(X, V)$ is a k_ω-space.*

COROLLARY. *Let X be a completely regular Hausdorff space and V a full variety. If Y is any finite subset of $F(X, V)$, then the topology induced on the subgroup generated by Y is discrete.*

If V is any β-variety and X is any completely regular Hausdorff space, then X is easily shown to be a closed subset of $F(X, V)$. However, the next theorem indicates that X is an open subset of $F(X, V)$ only in trivial cases.

THEOREM 32 ([51]). *Let X be a topological space and V a variety such that $F(X, V)$ exists. If X is an open subset of $F(X, V)$, then, providing $F(X, V)$ is not the Klein four-group, $F(X, V)$ has the discrete topology.*

We conclude this section by recording our lack of knowledge of topologically relatively free groups.

A topological group F is said to be *topologically relatively free with free generating space X* if X is a subspace of F which generates F algebraically and every continuous mapping of X into F can be extended to a continuous endomorphism of F.

PROBLEMS. If G is topologically relatively free, is the underlying group of G relatively free? **(P 1253)**

If G is topologically relatively free with free generating space X and G is algebraically relatively free with free generating set X, is G necessarily $F(X, V(G))$? **(P 1254)**

5. Topological laws. No doubt the reader has been waiting with bated breath to find out if varieties of topological groups can be described by "laws". An affirmative answer was recently given by Taylor [71]. The material in this section comes directly from Taylor's paper.

In accordance with Taylor, who has generalized the basic theory on varieties of topological groups to varieties of topological universal algebras, we present the work on laws in the context of topological universal algebras.

For convenience, fix a type of universal algebra and fix a class X of variables disjoint from all other sets and classes under consideration.

Let $(D, \leqslant)$ be an arbitrary directed set and $f: D \to \mathrm{Te}(X)$ a function from D to the class of terms over X of the given type. We will denote the term $f(d)$ by τ_d, and we let τ denote an arbitrary term. A *limit law* is an infinitary formal expression $f \to \tau$ with f and τ arbitrary, as above.

Generally, we do not mention D and simply write $\tau_d \to \tau$ to express "the net of terms τ_d always approaches the term τ". An assignment is a mapping $\theta\colon X \to \mathscr{A}$, where $\mathscr{A}$ is a topological algebra of the given type.

We say $\mathscr{A}$ *satisfies the limit law* $\tau_d \to \tau$ *under the assignment* θ if the net $\tau_d[\theta]$ converges to $\tau[\theta]$.

The topological algebra $\mathscr{A}$ is said to *obey the limit law* $\tau_d \to \tau$ if $\mathscr{A}$ satisfies the limit law $\tau_d \to \tau$ under every assignment $\theta\colon X \to \mathscr{A}$.

Example. $\Big[\big[\ldots[[x_1, x_2], x_3], \ldots\big], x_n\Big] \to 1$ with $[\cdot,\cdot]$ denoting commutator is a limit law which all nilpotent discrete topological groups obey.

For any class Γ of limit laws we denote by $\operatorname{Mod}\Gamma$ the class of all topological algebras $\mathscr{A}$ which obey every limit law in Γ.

THEOREM 33 ([71]). *If Γ is any class of limit laws, then* $\operatorname{Mod}\Gamma$ *is a wide variety of topological algebras.*

Recall that a wide variety is closed under subalgebras, Cartesian products, and continuous homomorphic images.

Example. Let Q be the group of rationals with the usual topology. Then Q/Z obeys the law $m!\,x \to 0$ but T does not. Thus T is not in the wide variety of topological groups generated by Q/Z.

THEOREM 34 ([71]). *Every wide variety of topological algebras is* $\operatorname{Mod}(\Sigma \cup \Gamma)$ *for some set Σ of algebraic laws and some class Γ of limit laws.*

Theorems 33 and 34 of Taylor show that wide varieties are characterized by limit laws. In an attempt to characterize varieties of topological algebras Taylor introduces the notion of contingent limit laws. The attempt is only partially successful in that, while he shows that every variety can be defined by a class of contingent limit laws (and algebraic laws), it has been pointed out by Dixon [15a] that not every class defined by contingent limit laws is a variety.

Take X and f as before. Now, let I be any set and, for each $i \in I$, let $(D_i, \leqslant)$ be a directed set. Assume that all x_d^i ($i \in I$ and $d \in D_i$) and x^i ($i \in I$) are distinct members of X. A *contingent limit law* is an infinitary formal expression λ of the form

$$\Big(\bigwedge_{i\in I} x_d^i \to x^i\Big) \Rightarrow (\tau_d \to \tau).$$

A topological algebra $\mathscr{A}$ *satisfies the contingent limit law λ under the assignment* $\theta\colon X \to \mathscr{A}$ if: the net $\tau_d[\theta]$ converges to $\tau[\theta]$ in the topology of $\mathscr{A}$ if, for each $i \in I$, $\theta(x_d^i)$ converges to $\theta(x^i)$ in the topology of $\mathscr{A}$.

Example. The contingent limit law $(x_d \to 1) \Rightarrow (z_d x_d z_d^{-1} \to 1)$, where d ranges over an arbitrary directed set D, is satisfied by all SIN-groups.

THEOREM 35 ([71]). *If V is any variety of topological algebras, then there exist a class of contingent limit laws and a set Σ of algebraic laws such that* $V = \operatorname{Mod}(\Sigma \cup \Gamma)$.

Any class defined by contingent limit laws is clearly closed under the operations S and C; the difficulty arises with Q. The next result provides some further information.

THEOREM 36 ([71]). *If λ is a contingent limit law of the form $(x_n \to x) \Rightarrow \tau_d \to \tau$ with n ranging over all natural numbers, and if $\mathscr{A}$ is a topological universal algebra satisfying λ and $\mathscr{A}$ is first countable, then λ holds in every open continuous homomorphic image of $\mathscr{A}$.*

COROLLARY ([71]). *If λ is a contingent law of the form $(x_n \to x) \Rightarrow \tau_d \to \tau$ with n ranging over all natural numbers, and Ω is a collection of first-countable algebras from some congruence-permutable variety such that $\mathscr{A}$ satisfies λ for every $\mathscr{A} \in \Omega$, then λ holds in every topological algebra in the variety of topological algebras generated by Ω.*

COROLLARY. *If λ is a contingent law of the form $(x_n \to x) \Rightarrow \tau_d \to \tau$ with n ranging over all natural numbers, and Ω is a collection of metrizable topological groups satisfying λ, then λ holds in every topological group in $V(\Omega)$.*

There are obviously numerous interesting questions one can ask about limit laws and contingent limit laws. We mention only four questions.

PROBLEMS. (i) What limit laws and contingent limit laws does R satisfy? (**P 1255**)

(ii) Find a collection of limit laws which characterize the wide variety of topological groups generated by R. (**P 1256**)

(iii) Find a collection of contingent limit laws which characterize the variety of topological groups generated by R. (**P 1257**)

(iv) Find a collection of contingent limit laws which characterize the variety of topological groups generated by all countable discrete groups. (**P 1258**)

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