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RECURSIVE PRIME MODELS FOR BOOLEAN ALGEBRAS

BY

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It follows from the work by Tarski [8] and Ershov [2] that if T is a complete extension of the theory T_B of Boolean algebra, then T is axiomatizable and atomistic (T has no atomless formulas). Thus, T has a recursive model [4] and T has a prime model. In this paper* we show that the prime model of T is finite-atomic (see Definition 1.2) and recursive, in fact strongly computable in the sense of Ershov (see [7]). The prime, computable Boolean algebras are shown to be the interval Boolean algebras of computable linear orders. From these results and the results of Hanf [3] we conclude that if T is a complete extension of T_B, then there is a finitely axiomatizable theory whose Lindenbaum algebra is the prime (countable finite-atomic) model of T.

1. Let $\mathscr{L}_{-1}$ denote the language of Boolean algebras with non-logical symbols $\cap$, $\cup$, ^{c}, 0, 1. The theory T_B of Boolean algebra has the usual axioms with the exception that we discard the axiom $0 \neq 1$. Thus, the Boolean algebra

$$\mathscr{M} = \langle M, \cap, \cup, {}^c, 0, 1\rangle$$

is trivial if and only if $0 = 1$; otherwise, $\mathscr{M}$ is non-trivial. Henceforth, we will use the same symbol to denote a structure and its universe.

If $\mathscr{M}$ is a structure for $\mathscr{L}_{-1}$, we make the usual definitions. For $a, b \in \mathscr{M}$,

$$a-b = a\cap b^c, \qquad a+b = (a-b)\cup(b-a),$$

a is *below* b if and only if $a \subseteq b$ if and only if $a\cap b = a$; $\mathscr{M}|_a$ is the Boolean algebra generated by $\{b \in \mathscr{M} \mid b \subseteq a\}$. $I(\mathscr{M})$ is the ideal of $\mathscr{M}$ generated by the set of all atomic elements and atomless elements. The Boolean algebra $\mathscr{M}_n$ is defined for each non-negative integer n as follows:

$$\mathscr{M}_0 = \mathscr{M}, \qquad \mathscr{M}_{n+1} = \mathscr{M}_n/I(\mathscr{M}_n)$$

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with $\bar{a} = \bar{b}$ in $\mathcal{M}_{n+1}$ if and only if $a+b \in I(\mathcal{M}_n)$. The *elementary characteristic of a Boolean algebra* $\mathcal{M}$, denoted by $\mathrm{EC}(\mathcal{M})$, is an ordered triple $\langle n, m, l\rangle$ such that

$$n = \begin{cases} \min\{r \mid \mathcal{M}_{r+1} \text{ is trivial}\} & \text{if it exists,} \\ \omega & \text{otherwise,} \end{cases}$$

$$m = \begin{cases} \sup\{r \mid \mathcal{M}_n \text{ has at least } r \text{ atoms}\} & \text{if } n < \omega, \\ 0 & \text{if } n = \omega, \end{cases}$$

$$l = \begin{cases} 0 & \text{if } n < \omega \text{ and } \mathcal{M}_n \text{ has no atomless elements or if } n = \omega, \\ 1 & \text{otherwise.} \end{cases}$$

We say $\mathcal{M}$ is an *r-level Boolean algebra* if $\mathrm{EC}(\mathcal{M}) = \langle n, m, l\rangle$ and $r \leqslant n$. The *elementary characteristic of an element* a, denoted by $\mathrm{EC}(a)$, is defined to be $\mathrm{EC}(\mathcal{M}|_a)$.

The following definition gives abbreviations for certain important formulas of $\mathcal{L}_{-1}$.

Definition 1.1. (i) $I_{-1}(x)$ is the formula $x = 0$.

(ii) $A_n(x)$ is the formula

$$\sim I_{n-1}(x) \wedge (y)\big[I_{n-1}\big((y \cap x)+y\big) \rightarrow I_{n-1}(x+y) \vee I_{n-1}(y)\big].$$

(iii) $AL_n(x)$ is the formula

$$\sim I_{n-1}(x) \wedge (y)\big[I_{n-1}\big((y \cap x)+y\big) \rightarrow \sim A_n(y)\big].$$

(iv) $AT_n(x)$ is the formula

$$\sim I_{n-1}(x) \wedge (y)\big[I_{n-1}\big((y \cap x)+y\big) \rightarrow \sim AL_n(y)\big].$$

(v) $I_n(x)$ is the formula

$$I_{n-1}(x) \vee AL_n(x) \vee AT_n(x) \vee (\exists y)(\exists z)\big[AL_n(y) \wedge AT_n(z) \wedge I_{n-1}\big(x+(y \cup z)\big)\big].$$

(vi) $x =_n y$ is the formula $I_n(x+y)$.

An element of a Boolean algebra which satisfies the formulas $I_n(x)$, $I_n(x) \wedge \sim I_{n-1}(x)$, $A_n(x)$, $AL_n(x)$ or $AT_n(x)$ is called an *n-level element*, an *n-element*, an *n-atom*, an *n-atomless element* or an *n-atomic element*, respectively.

We introduce new unary relation symbols $I^*_{-1}, I^*_n, A^*_n, AL^*_n, AT^*_n$ for each $n \geqslant 0$. New languages $\mathcal{L}_{i,j}$ for $i \geqslant -1$ and $0 \leqslant j \leqslant 3$ are defined inductively as follows:

For $n \geqslant 0$,

$$\mathscr{L}_{-1,0} = \mathscr{L}_{-1}, \quad \mathscr{L}_{n-1,1} = \mathscr{L}_{n-1,0} \cup \{I^*_{n-1}\},$$
$$\mathscr{L}_{n-1,2} = \mathscr{L}_{n-1,1} \cup \{A^*_n\}, \quad \mathscr{L}_{n-1,3} = \mathscr{L}_{n-1,2} \cup \{AL^*_n\},$$
$$\mathscr{L}_{n,0} = \mathscr{L}_{n-1,3} \cup \{AT^*_n\}, \quad \mathscr{L}_{\omega,0} = \bigcup_{0 \leqslant n < \omega} \mathscr{L}_{n,0}.$$

If T is a theory of Boolean algebras in $\mathscr{L}_{-1}$, then $T_{\mathscr{L}_{r,s}}$ is the theory in $\mathscr{L}_{r,s}$ obtained by adding — as axioms — the sentences equating each new unary relation in $\mathscr{L}_{r,s}$ with its corresponding formula of $\mathscr{L}_{-1}$.

Ershov [2] has shown that a theory T is a complete extension of T_B if there is an ordered triple $\langle n, m, l\rangle$ such that the models of T are precisely those Boolean algebras $\mathscr{M}$ for which $\mathrm{EC}(\mathscr{M}) = \langle n, m, l\rangle$; for convenience, we denote T by $\langle n, m, l\rangle$.

Definition 1.2. Let $\mathscr{M}$ be a model of $\langle n, m, l\rangle$. For $n < \omega$, $\mathscr{M}$ is *finite-atomic* if and only if

(i) for every $k < n$, every k-atomic element of $\mathscr{M}$ is the finite union of disjoint k-atoms;

(ii) for every n-atomic element a, either a or a^c has at most finitely many disjoint n-atoms below it.

For $n = \omega$, $\mathscr{M}$ is *finite-atomic* if and only if $\mathscr{M}$ satisfies (i) and

(iii) for every ω-element a there is an integer k such that a^c is a k-element.

The following results are useful (Lemma 1.1 is due to Ershov [2]).

LEMMA 1.1. *Let $\mathscr{M}$ be a Boolean algebra, n a non-negative integer, and $a \in \mathscr{M}$. Then*

(i) $\mathscr{M} \cong \mathscr{M}|_a \times \mathscr{M}|_{a^c}$;

(ii) $\mathscr{M}/I \cong (\mathscr{M}|_a)/I(\mathscr{M}|_a) \times (\mathscr{M}|_{a^c})/I(\mathscr{M}|_{a^c})$.

LEMMA 1.2. *Let $\mathscr{M}$ be a model of $\langle n, m, l\rangle_{\mathscr{L}_{r,i}}$, where $-1 \leqslant r < \omega$ and $0 \leqslant i \leqslant 3$. Let $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$ be sets of disjoint elements of $\mathscr{M}$ such that $\bigcup a_i = \bigcup b_i = 1$ and, for each $1 \leqslant j \leqslant s$, a_j and b_j satisfy the same unary relations of $\mathscr{L}_{r,i}$. If $\mathscr{N}_1$ and $\mathscr{N}_2$ are the subalgebras of $\mathscr{M}$ generated by $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$, respectively, and $h: \mathscr{N}_1 \to \mathscr{N}_2$ is defined by*

$$h\Big(\bigcup_{i=1} a_{k_i}\Big) = \bigcup_{i=1} b_{k_i},$$

then h is an $\mathscr{L}_{r,i}$-isomorphism.

LEMMA 1.3. *Let $\mathscr{M}$ be a finite-atomic Boolean algebra. Let $a, b \in \mathscr{M}$ be such that $\mathrm{EC}(a) = \mathrm{EC}(b)$. If $c_1, c_2 \in \mathscr{M}$, $a = c_1 \cup c_2$ and $c_1 \cap c_2 = 0$, then there are elements $d_1, d_2 \in \mathscr{M}$ such that*

$$b = d_1 \cup d_2, \quad d_1 \cap d_2 = 0, \quad \mathrm{EC}(d_1) = \mathrm{EC}(c_1) \quad \textit{and} \quad \mathrm{EC}(d_2) = \mathrm{EC}(c_2).$$

2. If L is a linear order with first element a_0 and last element b_0, then the *interval Boolean algebra* of L, denoted by $\mathscr{B}_L$, is the Boolean subalgebra of the power set of $[a_0, b_0)$ generated by $\{[x, y) \mid x, y \in L, x \leqslant y\}$. The Boolean algebra $\mathscr{B}_L$ has the following useful properties:

PROPOSITION 2.1. *Let L be a linear order with first and last elements.*

(i) *An element c is an atom of $\mathscr{B}_L$ if there are elements $a, b \in L$ such that b is the successor of a and $c = [a, b) = \{a\}$.*

(ii) *If $[a, b) \in \mathscr{B}_L$ is a dense linear suborder of L, then $[a, b)$ is an atomless element of $\mathscr{B}_L$.*

(iii) *Every $c \in \mathscr{B}_L$ has a unique representation as a union of intervals $[a_1, b_1), \ldots, [a_n, b_n)$, where $a_1 < b_1 < \ldots < a_n < b_n$.*

Example 1. Let L be the linear order

$$\{(1, 1), (2, y) \mid y \in Q'\},$$

where order is defined lexicographically and Q' is the linear order of the rational numbers with first element a_1 and last element a_2. Since $(2, a_1) \in L$ is the only successor in L and $[(2, a_1), (2, a_2))$ is a dense suborder of L, we see by Proposition 2.1 that $\mathrm{EC}(\mathscr{B}_L) = \langle 0, 1, 1\rangle$. Thus, $\mathscr{B}_L$ is countable and finite-atomic. Since Q' is a computable linear order, we see that $\mathrm{EC}(a)$ can be determined effectively. By the model completeness results of Ershov [2] we conclude that $\mathscr{B}_L$ is strongly computable (satisfaction in B_L is decidable).

Example 2. Let L^* be the linear order

$$\{(1, 1, a_1), (2, x, y), (2, a_2, a_2) \mid y \in Q'' \text{ if } a_1 < x < a_2, y \in L_0 \text{ if } x = a_1\},$$

where order is lexicographic, Q'' is the linear order obtained from Q' by adding the element a_0 as the predecessor of a_1, and L_0 is the linear order

$$\{(1/n, x) \mid x \in Q', n \text{ is a positive integer}\}$$

with lexicographic order. It follows easily from Proposition 2.1 that neither of the elements

$$a = [(1, 1, 1), (2, a_1, (1, a_2))) \quad \text{or} \quad b = [(2, a_1, (1, a_2)), (2, a_2, a_2))$$

is in $I(\mathscr{B}_{L^*})$; in fact, a is a 1-atom and b is a 1-atomless element. Thus, $\mathrm{EC}(\mathscr{B}_{L^*}) = \langle 1, 1, 1\rangle$ and, as in the case of $\mathscr{B}_L$, $\mathscr{B}_{L^*}$ is countable and finite-atomic. Since Q'' and L_0 are computable linear orders, it follows, as in Example 1, that $\mathscr{B}_{L^*}$ is strongly computable.

It should be noted that L^* can be obtained from L in the following way: Replace each successor in L by a suborder isomorphic to L_0 and replace each element in a dense suborder of L by a suborder isomorphic to Q''. The effect of this process is to change each 0-atom of $\mathscr{B}_L$ to a 1-atom in $\mathscr{B}_{L^*}$ and each 0-atomless element in $\mathscr{B}_L$ to a 1-atomless element in $\mathscr{B}_{L^*}$.

By this process we will obtain countable, finite-atomic, strongly computable Boolean algebras of each elementary characteristic.

3. Definition 3.1. Let L be a linear order. L is a *special linear order* if and only if L satisfies the following conditions:

(i) If $b < d$ in L and b is a successor, then there are elements $e, f \in L$ such that $b < e < f < d$ and $[e, f)$ is a dense linear suborder of L.

(ii) The first (last) element of L, if it exists, is not a predecessor (successor).

Elements a and b in L form a *predecessor-successor pair* (p-s pair) if $a < b$ and $[a, b) = \{a\}$.

We recall that if L and K are linear orders, then $L+K$ is the linear order on $L \cup K$ which extends L and K so that all elements of L precede all elements of K. The following results are clear from Definition 3.1.

Lemma 3.1. (i) *If L and K are special linear orders, then so is $L+K$.*

(ii) *If L is a special linear order, $a, b \in L$ and $[a, b)$ is non-empty, then*

(a) *$[a, b)$ is finite if and only if $[a, b) = \{a\}$,*

(b) *$[a, b)$ is infinite if and only if there are elements $a \leqslant c < d \leqslant b$ such that $[c, d)$ is a dense linear order.*

For each linear order L we define the following partition of L:

$\Delta_L^1 = \{x \mid x$ is a predecessor or a successor and x is not the first or the last element of $L\}$,

$\Delta_L^2 = \{x \mid x \notin \Delta_L^1$ and x is neither the first nor the last element of $L\}$,

$\Delta_L^3 = \{x \mid x$ is the first or the last element of $L\}$.

We use this partition to define a new linear order L^* called the specialization of L.

Definition 3.2. Let L be a linear order. We define the *specialization* of L, denoted by L^*, to be the linear order

$$\{(a, x), (b, y), (c, c) \mid a \in \Delta_L^1 \text{ and } x \in L_0,\ b \in \Delta_L^2 \text{ and } y \in Q'',\ c \in \Delta_L^3\},$$

where order is lexicographic.

This definition yields the following result which allows us to define special linear orders inductively.

Proposition 3.1. *Let L be a linear order. If L is countable, special, or if L has the first (last) element, then L^* is countable, special, or has the first (last) element, respectively.*

The following result is the tool needed to define the countable, finite-atomic Boolean algebra for each theory $\langle n, m, l\rangle$.

Theorem 3.1. *Let L be a special linear order with first and last elements. There is an epimorphism $h: \mathscr{B}_{L^*} \to \mathscr{B}_L$ such that* $\ker(h) = I(\mathscr{B}_{L^*})$. *Hence*

$$\mathscr{B}_{L^*}/\ker(h) \cong \mathscr{B}_L,$$

and if $\mathrm{EC}(\mathscr{B}_L) = \langle n, m, l\rangle$, *then* $\mathrm{EC}(\mathscr{B}_{L^*}) = \langle n+1, m, l\rangle$ *for* $1 \leqslant n < \omega$.

Proof. We define $h : \mathscr{B}_{L^*} \to \mathscr{B}_L$ as follows: If $c \in \mathscr{B}_{L^*}$ and

$$\bigcup_{i=1}^{n} \left[(a_{i,1}, a_{i,2}), (b_{i,1}, b_{i,2})\right)$$

is its unique representation, then

$$h(c) = \bigcup_{i=1}^{n} [a_{i,1}, b_{i,1}).$$

It follows from Proposition 2.1 (iii) and Definition 3.2 that h is an epimorphism.

To prove that $\ker(h) \subseteq I(\mathscr{B}_{L^*})$ let $c \in \ker(h)$; it is sufficient to assume that $c = \left[(a_1, a_2), (b_1, b_2)\right)$. By the definition of h, $c \in \ker(h)$ if and only if $a_1 = b_1$. Since a_1 cannot be the first element of L, $a_1 \in \Delta_L^1$ or $a_1 \in \Delta_L^2$. If $a_1 \in \Delta_L^1$, then there are integers n and m such that $a_2 = (1/n, x)$ and $b_2 = (1/m, y)$. By the definition of L^*, c can have at most finitely many p-s pairs in L^*. Thus, c has at most finitely many atoms below it (Proposition 2.1 (i)) and $c \in I(\mathscr{B}_{L^*})$. If $a_1 \in \Delta_L^2$, then $\left[(a_1, a_2), (b_1, b_2)\right)$ is order isomorphic to a subinterval of Q''. In this case c has at most one p-s pair and, as in the previous case, $c \in I(\mathscr{B}_{L^*})$.

Let $c = \left[(a_1, a_2), (b_1, b_2)\right) \in I(\mathscr{B}_{L^*})$. It follows from Lemma 3.1 (ii) and Proposition 2.1 that $\mathscr{B}_{L^*}$ has no atomic element with infinitely many atoms below it. By Definition 3.2 it is clear that $a_1 = b_1$. Since $h(c) = [a_1, b_1)$ and $a_1 = b_1$, we have $h(c) = 0$. Therefore, $c \in \ker(h)$. We conclude that $\ker(h) = I(\mathscr{B}_{L^*})$.

It now follows that $\mathscr{B}_{L^*}/\ker(h) \cong \mathscr{B}_L$. Since $\ker(h) = I(\mathscr{B}_{L^*})$,

$$\mathscr{B}_{L^*}/I(\mathscr{B}_{L^*}) \cong \mathscr{B}_L.$$

Thus, if $\mathrm{EC}(\mathscr{B}_L) = \langle n, m, l\rangle$ and $0 \leqslant n < \omega$, then

$$\mathrm{EC}(\mathscr{B}_{L^*}) = \langle n+1, m, l\rangle.$$

The next lemma follows easily from Definition 3.2, Proposition 2.1 (iii) and Lemma 1.1.

LEMMA 3.2. *Let L be a special linear order with first and last elements.*

(i) *If L is countable, then so is $\mathscr{B}_{L^*}$.*

(ii) *If $\mathscr{B}_L$ is finite-atomic, then so is $\mathscr{B}_{L^*}$.*

Definition 3.3. Let L be a linear order and n a positive integer. The *linear order L^{n^*}* is defined inductively as follows:

(i) $L^{1^*} = L^*$,

(ii) $L^{(n+1)^*} = (L^{n^*})^*$.

COROLLARY 3.1. *Let s be a positive integer and let L be a countable, special linear order such that $\mathscr{B}_L$ is finite-atomic and $\mathrm{EC}(\mathscr{B}_L) = \langle n, m, l\rangle$. Then L^{s^*} is a countable, special linear order and $\mathscr{B}_{L^{s^*}}$ is countable, finite-atomic and*

$$\mathrm{EC}(\mathscr{B}_{L^{s^*}}) = \langle n+s, m, l\rangle.$$

4. We are now in a position to define, for each theory $\langle n, m, l\rangle$, a countable, finite-atomic, strongly computable Boolean algebra, which we will denote by $\mathscr{B}_{n,m,l}$. We define $\mathscr{B}_{n,m,l}$ to be the interval Boolean algebra for a countable, special linear order $L_{n,m,l}$ determined as follows:

(i) For $1 \leqslant m \leqslant \omega$, $L_{0,m,0}$ is $\{r \mid 0 \leqslant r \leqslant m,\ r \text{ an integer}\}$ with the usual order.

(ii) For $0 \leqslant m < \omega$, $L_{0,m,1}$ is the linear order

$$\{(r, x) \mid 0 \leqslant r \leqslant m+1,\ r \text{ an integer},\ x \in Q'\}$$

with lexicographic ordering. $L_{n,m,1} = (L_{0,m,1})^{n^*}$ for $0 < n < \omega$.

(iii) $L_{0,\omega,1}$ is the linear order

$$\{(r, 0), (\omega, x) \mid 1 \leqslant r < \omega,\ r \text{ an integer},\ x \in Q'\}$$

with order defined lexicographically.

(iv) For $0 \leqslant m \leqslant \omega$, $L_{1,m,0}$ is the linear order

$$\{(0, 0, 0), (r, 1/s, x) \mid 0 \leqslant r \leqslant m,\ 1 < s < \omega,\ r \text{ and } s \text{ are integers},\ x \in Q'\}$$

with lexicographic ordering. For $1 < n < \omega$ we put

$$L_{n,m,0} = (L_{1,m,0})^{(n-1)^*}.$$

(v) We put

$$L_{1,\omega,1} = L_{1,\omega,0} + (L_{0,0,1})^*$$

and

$$L_{n,\omega,1} = (L_{1,\omega,1})^{(n-1)^*} \quad \text{for } 1 < n < \omega.$$

Since Q', Q'' and L_0 are computable linear orders, each linear order $L_{n,m,l}$ defined above is computable. It is also clear that if $a \in \mathscr{B}_{n,m,l}$, then $\mathrm{EC}(a)$ can be determined effectively. We conclude that $\mathscr{B}_{n,m,l}$ is strongly computable. By Lemma 3.2 and Corollary 3.1, $\mathscr{B}_{n,m,l}$ is countable, finite-atomic and has the elementary characteristic $\langle n, m, l\rangle$.

Finally, we define $\mathscr{B}_{\omega,0,0}$ to be the weak direct product of

$$\{\mathscr{B}_{n,1,0} \mid n \text{ a non-negative integer}\}.$$

The fact that $\mathscr{B}_{\omega,0,0}$ is countable and finite-atomic is clear. Since each $\mathscr{B}_{n,1,0}$ is strongly computable and since each element of $\mathscr{B}_{\omega,0,0}$ has only finitely many non-zero coordinates, we see that $\mathscr{B}_{\omega,0,0}$ is strongly computable.

We summarize the results of this section in the following:

THEOREM 4.1. *For each elementary characteristic $\langle n, m, l\rangle$ the Boolean algebra $\mathscr{B}_{n,m,l}$ is countable, finite-atomic and strongly computable.*

5. In this section we show that the atomic models of the theory $\langle n, m, l\rangle$ are precisely finite-atomic.

LEMMA 5.1. *Let $\mathscr{M}$ be a finite-atomic model of $\langle n, m, l\rangle_{\mathscr{L}_{n,0}}$. Let $\{a_1, \ldots, a_s\}$ and $\{b_1, \ldots, b_s\}$ be sets of disjoint elements of $\mathscr{M}$ such that $\bigcup a_i = \bigcup b_i = 1$ and $\mathrm{EC}(a_i) = \mathrm{EC}(b_i)$ for each $i \in \{1, \ldots, s\}$. Then $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_s)$ satisfy the same formulas of $\mathscr{L}_{n,0}$ in $\mathscr{M}$.*

Proof. Clearly, a_i and b_i satisfy the same unary relations of $\mathscr{L}_{n,0}$ in $\mathscr{M}$. The proof is completed by induction on formula complexity using Lemma 1.2. The only difficult step is that involving the existential quantifier.

Suppose that $\psi(x_1, \ldots, x_s)$ is the formula

$$(\exists x_0)\gamma(x_0, \ldots, x_s)$$

and $a_0 \in \mathscr{M}$ is such that $(a_0, \ldots, a_s)$ satisfies γ in $\mathscr{M}$. Let $c_1, \ldots, c_v$ be the atoms of the subalgebra of $\mathscr{M}$ generated by $a_0, \ldots, a_s$. For each $i \in \{0, \ldots, s\}$ there is a term $\sigma_i(x_1, \ldots, x_v)$ of $\mathscr{L}_{n,0}$ such that $a_i = \sigma_i(c_1, \ldots, c_v)$. Using Lemma 1.3 choose elements $d_1, \ldots, d_v$ such that

$$\mathrm{EC}(d_i) = \mathrm{EC}(c_i) \text{ for } i \in \{1, \ldots, v\} \quad \text{and} \quad b_i = \sigma_i(d_1, \ldots, d_v).$$

Since $(c_1, \ldots, c_v)$ satisfies

$$\gamma\big(\sigma_0(x_1, \ldots, x_v), \ldots, \sigma_s(x_1, \ldots, x_v)\big) \quad \text{in } \mathscr{M},$$

$(d_1, \ldots, d_v)$ satisfies the same formula in $\mathscr{M}$ by the inductive hypothesis. Thus, letting

$$b_0 = \sigma_0(d_1, \ldots, d_v),$$

$(b_0, \ldots, b_s)$ satisfies $\gamma(x_0, \ldots, x_s)$ in $\mathscr{M}$ and $(b_1, \ldots, b_s)$ satisfies the formula $(\exists x_0)\gamma(x_0, \ldots, x_s)$ in $\mathscr{M}$. The converse is proved similarly.

THEOREM 5.1. *Let T be the theory $\langle n, m, l\rangle$ and $\mathscr{M}$ a model of T. Then $\mathscr{M}$ is an atomic model of T if and only if $\mathscr{M}$ is finite-atomic.*

Proof. If $\mathscr{M}$ is not finite-atomic, then there is an element $b \in \mathscr{M}$, which is k-atomic for some $k \leqslant n$, such that b and b^c have infinitely many k-atoms. Then b satisfies a non-principal 1-type of T. Thus, $\mathscr{M}$ is not an atomic model of T.

Suppose that $\mathscr{M}$ is finite-atomic and $(a_1, \ldots, a_s)$ is an s-tuple of elements of $\mathscr{M}$. Let p be the s-type of T satisfied by $(a_1, \ldots, a_s)$. We show that p is a principal s-type.

Let Λ be the set of all terms $\tau(x_1, \ldots, x_s)$ in $\mathscr{L}_{-1}$ of the form

$$\bigcup_{i=1}^{v}\sigma_i(x_{k_i}) \quad \text{for } v \in \{1, \ldots, s\} \text{ and } 1 \leqslant k_1 < \ldots < k_v \leqslant s,$$

where $\sigma_i(x)$ is either x or x^c. Let $\tau_1, \ldots, \tau_r$ be an enumeration of Λ. Since $\mathscr{M}$ is finite-atomic, it is clear that there is a formula $\varphi_i(x_1, \ldots, x_s)$ which specifies the elementary characteristic of $\tau_i(a_1, \ldots, a_s)$ (or, if necessary, $\tau_i(a_1, \ldots, a_s)^c$). Let $\Omega(x_1, \ldots, x_s)$ be the formula

$$\varphi_1(x_1, \ldots, x_s) \wedge \ldots \wedge \varphi_r(x_1, \ldots, x_s).$$

If $(b_1, \ldots, b_s)$ is an s-tuple of elements of $\mathcal{M}$ satisfying $\Omega(x_1, \ldots, x_s)$, then it follows from the definition of $T_{\mathcal{L}_{n,0}}$ and from Lemma 5.1 that $(a_1, \ldots, a_s)$ and $(b_1, \ldots, b_s)$ satisfy the same formulas of $\mathcal{L}_{-1}$. Thus, by the completeness of T, p is a principal s-type generated by $\Omega(x_1, \ldots, x_s)$. We conclude that $\mathcal{M}$ is an atomic model of T.

COROLLARY 5.1. *The prime model of* $\langle n, m, l\rangle$ *is* $\mathcal{B}_{n,m,l}$.

COROLLARY 5.2. *For each* $\mathcal{B}_{n,m,l}$ *there is a finitely axiomatizable theory whose Lindenbaum algebra is isomorphic to* $\mathcal{B}_{n,m,l}$.

Proof. Hanf [3] has shown that if B is a recursive Boolean algebra, then there is a finitely axiomatizable theory whose Lindenbaum algebra is isomorphic to $\mathcal{B}$. It is clear from the definition of $\mathcal{B}_{n,m,l}$ that since $L_{n,m,l}$ is computable, a Boolean algebra $\mathcal{B}$ can be defined so that $\mathcal{B} \cong \mathcal{B}_{n,m,l}$, the universe of $\mathcal{B}$ and the relations $\cap$, $\cup$, ^{c} are recursive. Thus, there is a finitely axiomatizable theory whose Lindenbaum algebra is isomorphic to $\mathcal{B}_{n,m,l}$.

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