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OPTIMAL BOUNDS FOR EXPONENTIAL SUMS IN TERMS OF DISCREPANCY

BY

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1. Introduction. Let $x_1, \ldots, x_N$ be real numbers and let

$$D_N^* = D_N^*(x_1, \ldots, x_N) = \sup_{0<t\leqslant 1} \left|\frac{1}{N}\sum_{n=1}^{N} \chi_t(\{x_n\})-t\right|$$

be their discrepancy, where χ_t denotes the characteristic function of the interval $[0, t)$ and $\{u\} = u - \lfloor u \rfloor$ is the fractional part of the real number u. It is well known that the exponential sum

$$\sum_{n=1}^{N} e(x_n) \quad \text{with } e(u) = e^{2\pi i u}$$

can be bounded in terms of D_N^*. It was shown by van der Corput and Pisot [3] that

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| \leqslant 2\pi D_N^*.$$

In the book of Kuipers and Niederreiter [1], p. 143, this was improved to

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| \leqslant 4D_N^*. \tag{1}$$

In fact, 4 is the least constant such that (1) is valid for all N and for all $x_1, \ldots, x_N$ (see [1], p. 163, Exercise 5.34).

This still leaves open the possibility of replacing 4 by a smaller constant that may depend on N. For fixed N, the least constant c_N such that

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| \leqslant c_N D_N^*$$

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holds for all $x_1, \ldots, x_N$ is given by

$$c_N = \sup_{x_1,\ldots,x_N} \frac{1}{ND_N^*(x_1, \ldots, x_N)} \left| \sum_{n=1}^{N} e(x_n) \right|. \tag{2}$$

The problem of determining c_N was raised by Kuipers and Niederreiter [1]. From (1) and the remark following it we get

$$c_N \leqslant 4 \quad \text{and} \quad \lim_{N\to\infty} c_N = 4.$$

The example $x_1 = \ldots = x_N = \frac{1}{2}$ shows that $c_N \geqslant 2$. The only known values of c_N are $c_1 = 2$ and

$$c_2 = \max_{0 \leqslant x \leqslant 1/2} \frac{4 \sin \pi x}{1+2x} = 2.11\ldots$$

(see [1], p. 160, Exercises 5.8 and 5.9). The problem of determining the exact value of c_N for large N appears to be quite difficult. In this paper we establish lower and upper bounds for c_N which show that $4-c_N$ is of the order of magnitude $N^{-2/3}$.

THEOREM. *For all $N \geqslant 3$ we have*

$$4-10N^{-2/3} \leqslant c_N \leqslant 4-2N^{-2/3}.$$

It will become apparent in the proofs that the coefficients 10, respectively 2, of $N^{-2/3}$ can be improved for sufficiently large N.

A question related to that of determining c_N was studied by Montgomery and Niederreiter [2]. Some of the results of this work are useful for establishing the upper bound in our theorem. The problem considered in [2] is to determine, for fixed $N \geqslant 2$, the least constant d_N such that

$$\left| \frac{1}{N} \sum_{n=1}^{N} e\left(\frac{n}{N}+\theta_n\right) \right| \leqslant d_N K \tag{3}$$

holds for any $0 \leqslant K \leqslant \frac{1}{2}$ and any N numbers $\theta_1, \ldots, \theta_N$ with $|\theta_n| \leqslant K$ for $1 \leqslant n \leqslant N$. The following formula was obtained in that paper:

$$d_N = \begin{cases} \dfrac{4\pi}{N\sin(\pi/N)} & \text{for } N \text{ even,} \\[2ex] \dfrac{2\pi}{N\sin(\pi/2N)} & \text{for } N \text{ odd.} \end{cases} \tag{4}$$

The present paper is organized as follows. In Section 2 we prove the lower bound for c_N, and in Section 3 the upper bound. In Section 4 we describe an alternative method of obtaining an upper bound for c_N. This method yields a somewhat weaker result, but it does not rely on the work of Montgomery and Niederreiter [2] and it contains some features that may be of independent interest.

2. The lower bound. Since $c_N \geqslant 2$ for all N and since $4-10N^{-2/3} \leqslant 2$ for $N \leqslant 11$, we can assume $N \geqslant 12$. We recall the following explicit formula for D_N^* given in [1] (p. 91): if

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N < 1,$$

then

$$D_N^* = \frac{1}{2N} + \max_{1 \leqslant n \leqslant N} \left| x_n - \frac{2n-1}{2N} \right|. \tag{5}$$

We first consider the case where N is even. Let k be an integer with $1 \leqslant k \leqslant N/2$ that will be determined later, and let $0 < \varepsilon < 1/(2N)$. Put

$$x_n = \begin{cases} 0 & \text{for } 1 \leqslant n \leqslant k, \\ (n-k)/N & \text{for } k+1 \leqslant n \leqslant N/2, \\ (n+k-1)/N & \text{for } N/2+1 \leqslant n \leqslant N-k, \\ 1-\varepsilon & \text{for } N-k+1 \leqslant n \leqslant N. \end{cases}$$

Using (5), it is easily seen that

$$D_N^* = k/N. \tag{6}$$

Furthermore,

$$\begin{aligned} \sum_{n=1}^{N} e(x_n) &= k + \sum_{n=1}^{(N/2)-k} e\left(\frac{n}{N}\right) + \sum_{n=1}^{(N/2)-k} e\left(-\frac{n}{N}\right) + ke(-\varepsilon) \\ &= k + ke(-\varepsilon) + 2\sum_{n=1}^{(N/2)-k} \cos\frac{2\pi n}{N}, \end{aligned}$$

and so (2) and (6) yield

$$c_N \geqslant \frac{1}{k}\left| k + ke(-\varepsilon) + 2\sum_{n=1}^{(N/2)-k} \cos\frac{2\pi n}{N} \right|.$$

Letting $\varepsilon \to 0+$, we get

$$c_N \geqslant 2 + \frac{2}{k}\sum_{n=1}^{(N/2)-k} \cos\frac{2\pi n}{N}.$$

Standard trigonometric identities yield

$$\sum_{n=1}^{(N/2)-k} \cos\frac{2\pi n}{N} = \frac{\sin((2k-1)\pi/N) - \sin(\pi/N)}{2\sin(\pi/N)},$$

and so

$$c_N \geq 2 + \frac{\sin((2k-1)\pi/N)}{k\sin(\pi/N)} - \frac{1}{k}.$$

Using $\sin x \geq x - \frac{1}{6}x^3$ for $0 \leq x \leq \pi$, we obtain

$$c_N \geq 2 + \frac{2\pi}{N\sin(\pi/N)} - \frac{1}{k}\left(\frac{\pi}{N\sin(\pi/N)} + 1\right) - \frac{4\pi^3 k^2}{3N^3 \sin(\pi/N)}. \tag{7}$$

Now choose

$$k = \left\lceil \frac{N}{2\pi}\left(\frac{3\pi}{N} + 3\sin\frac{\pi}{N}\right)^{1/3} \right\rceil,$$

where $\lceil u \rceil$ denotes the least integer $\geq u$. It is easily seen that $1 \leq k \leq N/2$ for $N \geq 12$. Since

$$k \leq \frac{N}{2\pi}\left(\frac{3\pi}{N} + 3\sin\frac{\pi}{N}\right)^{1/3} + 1,$$

we get from (7) after some simplifications

$$\begin{aligned}
c_N &\geq 2 + \frac{2\pi}{N\sin(\pi/N)} - \frac{\pi}{N\sin(\pi/N)}\left(\frac{3\pi}{N} + 3\sin\frac{\pi}{N}\right)^{2/3} \\
&\quad - \frac{4\pi^2}{3N^2\sin(\pi/N)}\left(\frac{3\pi}{N} + 3\sin\frac{\pi}{N}\right)^{1/3} - \frac{4\pi^3}{3N^3\sin(\pi/N)} \\
&\geq 2 + \frac{2\pi}{N\sin(\pi/N)} - \frac{\pi}{N\sin(\pi/N)}\left(\frac{6\pi}{N}\right)^{2/3} \\
&\quad - \frac{4\pi^2}{3N^2\sin(\pi/N)}\left(\frac{6\pi}{N}\right)^{1/3} - \frac{4\pi^3}{3N^3\sin(\pi/N)} \\
&= 2 + \frac{\pi}{N\sin(\pi/N)}\left(2 - (6\pi)^{2/3}N^{-2/3} - \frac{4\pi}{3}(6\pi)^{1/3}N^{-4/3} - \frac{4\pi^2}{3}N^{-2}\right) \\
&\geq 2 + \frac{\pi}{N\sin(\pi/N)}\left(2 - (6\pi)^{2/3}N^{-2/3} - \frac{4\pi}{3}(6\pi)^{1/3}(12)^{-2/3}N^{-2/3}\right. \\
&\quad \left. - \frac{4\pi^2}{3}(12)^{-4/3}N^{-2/3}\right) \\
&\geq 2 + \frac{\pi}{N\sin(\pi/N)}(2 - 10N^{-2/3}) \geq 4 - 10N^{-2/3}
\end{aligned}$$

for $N \geq 12$. This proves the lower bound for even N.

For odd N we let k be an integer with $1 \leqslant k \leqslant (N-1)/2$ and let $0 < \varepsilon < 1/(2N)$. Put $x_{(N+1)/2} = 1/2$ and

$$x_n = \begin{cases} 0 & \text{for } 1 \leqslant n \leqslant k, \\ (n-k)/N & \text{for } k+1 \leqslant n \leqslant (N-1)/2, \\ (n+k-1)/N & \text{for } (N+3)/2 \leqslant n \leqslant N-k, \\ 1-\varepsilon & \text{for } N-k+1 \leqslant n \leqslant N. \end{cases}$$

From (5) we get again $D_N^* = k/N$. Furthermore

$$\sum_{n=1}^{N} e(x_n) = k + \sum_{n=1}^{((N-1)/2)-k} e\left(\frac{n}{N}\right) - 1 + \sum_{n=1}^{((N-1)/2)-k} e\left(-\frac{n}{N}\right) + ke(-\varepsilon)$$

$$= k - 1 + ke(-\varepsilon) + 2 \sum_{n=1}^{((N-1)/2)-k} \cos\frac{2\pi n}{N},$$

and so, with $\varepsilon \to 0+$,

$$c_N \geqslant 2 - \frac{1}{k} + \frac{2}{k} \sum_{n=1}^{((N-1)/2)-k} \cos\frac{2\pi n}{N}.$$

Now

$$\sum_{n=1}^{((N-1)/2)-k} \cos\frac{2\pi n}{N} = \frac{\sin(2k\pi/N) - \sin(\pi/N)}{2\sin(\pi/N)},$$

and so

$$c_N \geqslant 2 + \frac{\sin(2k\pi/N)}{k\sin(\pi/N)} - \frac{2}{k}.$$

Using $\sin x \geqslant x - \frac{1}{6}x^3$ for $0 \leqslant x \leqslant \pi$, we obtain

$$c_N \geqslant 2 + \frac{2\pi}{N\sin(\pi/N)} - \frac{4\pi^3 k^2}{3N^3 \sin(\pi/N)} - \frac{2}{k}$$

$$\geqslant 2 + \frac{2\pi}{N\sin(\pi/N)} - \frac{4\pi^3 k^2}{3N^3 \sin(\pi/N)} - \frac{1}{k}\left(\frac{\pi}{N\sin(\pi/N)} + 1\right).$$

This is the same lower bound as in (7). We choose k as before and note that $1 \leqslant k \leqslant (N-1)/2$ for $N \geqslant 12$. The earlier calculation yields then the desired lower bound for odd N.

3. The upper bound. Since both D_N^* and the exponential sum in (2) only depend on the fractional parts of the numbers $x_1, \ldots, x_N$ and since the order of the terms is irrelevant, we can assume that

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N < 1.$$

Put

$$\theta_n = x_n - \frac{2n-1}{2N} \quad \text{for } 1 \leqslant n \leqslant N$$

and note that (5) implies

$$(8) \qquad |\theta_n| \leqslant D_N^* - \frac{1}{2N} =: K \quad \text{for } 1 \leqslant n \leqslant N.$$

Also

$$(9) \qquad \left|\sum_{n=1}^{N} e(x_n)\right| = \left|\sum_{n=1}^{N} e\left(\theta_n + \frac{n}{N} - \frac{1}{2N}\right)\right| = \left|\sum_{n=1}^{N} e\left(\frac{n}{N} + \theta_n\right)\right|.$$

We distinguish now various cases depending on the values of D_N^* and N. In the first case we consider $D_N^* \leqslant \frac{1}{2} N^{-1/3}$. Then

$$K = D_N^* - \frac{1}{2N} \leqslant (1 - N^{-2/3}) D_N^*,$$

and so (3), (4), (8), and (9) yield

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| = \left|\frac{1}{N}\sum_{n=1}^{N} e\left(\frac{n}{N} + \theta_n\right)\right| \leqslant \frac{4\pi}{N\sin(\pi/N)}(1 - N^{-2/3}) D_N^*.$$

Using $\sin x \geqslant x - \frac{1}{6}x^3$ for $0 \leqslant x \leqslant \pi$, we get

$$\frac{1}{ND_N^*}\left|\sum_{n=1}^{N} e(x_n)\right| \leqslant \left(1 - \frac{\pi^2}{6N^2}\right)^{-1} (4 - 4N^{-2/3}).$$

Now

$$(4 - 2N^{-2/3})\left(1 - \frac{\pi^2}{6N^2}\right) \geqslant 4 - 2N^{-2/3} - \frac{2\pi^2}{3N^2}$$

$$\geqslant 4 - 2N^{-2/3} - \frac{2\pi^2}{3} 3^{-4/3} N^{-2/3} \geqslant 4 - 4N^{-2/3}$$

for $N \geqslant 3$, and so

$$\frac{1}{ND_N^*}\left|\sum_{n=1}^{N} e(x_n)\right| \leqslant 4 - 2N^{-2/3}.$$

The following argument applies whenever $D_N^* \leqslant \frac{1}{2}$. If N is even, then by (9) and a result of Montgomery and Niederreiter [2] we have

$$\left|\sum_{n=1}^{N} e(x_n)\right| \leqslant F_K\left(\frac{1}{2N}\right) := 2\left\lfloor KN + \frac{1}{2}\right\rfloor$$

$$+\frac{1}{\sin(\pi/N)}\left(\sin 2\pi K+\sin\frac{\pi}{N}\left(2\left\{KN+\frac{1}{2}\right\}-1\right)\right)$$

$$=2\lfloor ND_N^*\rfloor+\frac{1}{\sin(\pi/N)}\left(\sin 2\pi\left(D_N^*-\frac{1}{2N}\right)+\sin\frac{\pi}{N}(2\{ND_N^*\}-1)\right)$$

$$=2ND_N^*+\frac{\sin 2\pi(D_N^*-(2N)^{-1})}{\sin(\pi/N)}+\frac{\sin\pi N^{-1}(2\{ND_N^*\}-1)}{\sin(\pi/N)}-2\{ND_N^*\}.$$

Now consider the function

$$g(x)=\frac{\sin\pi N^{-1}(2x-1)}{\sin(\pi/N)}-2x \quad \text{for } 0\leqslant x\leqslant 1.$$

For $0\leqslant x\leqslant\frac{1}{2}$ it is clear that $g(x)\leqslant 0$. For $\frac{1}{2}<x\leqslant 1$ we have

$$g(x)\leqslant\frac{\pi(2x-1)}{N\sin(\pi/N)}-2x\leqslant\frac{\pi}{2}(2x-1)-2x=(\pi-2)x-\frac{\pi}{2}\leqslant 0.$$

Therefore

$$\left|\sum_{n=1}^{N}e(x_n)\right|\leqslant 2ND_N^*+\frac{\sin 2\pi(D_N^*-(2N)^{-1})}{\sin(\pi/N)}. \tag{10}$$

If N is odd and $K>(4N)^{-1}$, then by (9) and a result of Montgomery and Niederreiter [2] we have

$$\left|\sum_{n=1}^{N}e(x_n)\right|\leqslant F_K(0):=2\lfloor KN\rfloor+1$$

$$+\frac{1}{\sin(\pi/N)}\left(\sin 2\pi K+\sin\frac{\pi}{N}(2\{KN\}-1)\right)$$

$$=2\left\lfloor ND_N^*-\frac{1}{2}\right\rfloor+1$$

$$+\frac{1}{\sin(\pi/N)}\left(\sin 2\pi\left(D_N^*-\frac{1}{2N}\right)+\sin\frac{\pi}{N}\left(2\left\{ND_N^*-\frac{1}{2}\right\}-1\right)\right)$$

$$=2ND_N^*+\frac{\sin 2\pi(D_N^*-(2N)^{-1})}{\sin(\pi/N)}+g\left(\left\{ND_N^*-\frac{1}{2}\right\}\right).$$

Using $g(x)\leqslant 0$ for $0\leqslant x\leqslant 1$, we get again (10). Therefore (10) holds whenever $D_N^*\leqslant\frac{1}{2}$, with the additional condition $K>(4N)^{-1}$ for odd N.

In the second case we consider $\frac{1}{2}N^{-1/3}<D_N^*\leqslant\frac{1}{4}$. Note that this case applies only for $N\geqslant 9$ and that the additional condition $K>(4N)^{-1}$ for odd

N in (10) is automatically satisfied. It follows from (10) that

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 2+\frac{\sin 2\pi D_N^*}{ND_N^*\sin(\pi/N)}.$$

Since the function $(\sin 2\pi x)/x$, $0 < x \leqslant \frac{1}{4}$, is decreasing, we obtain

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 2+\frac{2\sin \pi N^{-1/3}}{N^{2/3}\sin(\pi/N)}. \tag{11}$$

Using

$$x-\frac{1}{6}x^3 \leqslant \sin x \leqslant x-\frac{1}{6}x^3+\frac{1}{120}x^5 \quad \text{for } 0 \leqslant x \leqslant \pi,$$

we get

$$\frac{\sin \pi N^{-1/3}}{\sin(\pi/N)} \leqslant \frac{N^{2/3}-(\pi^2/6)+(\pi^4/120)N^{-2/3}}{1-(\pi^2/6)N^{-2}}.$$

Now

$$(N^{2/3}-1)\left(1-\frac{\pi^2}{6}N^{-2}\right)-\frac{\pi^4}{120}N^{-2/3} \geqslant N^{2/3}-1-\frac{\pi^2}{6}N^{-4/3}-\frac{\pi^4}{120}N^{-2/3}$$

$$\geqslant N^{2/3}-1-\frac{\pi^2}{6}9^{-4/3}-\frac{\pi^4}{120}9^{-2/3} \geqslant N^{2/3}-\frac{\pi^2}{6},$$

and so

$$\frac{\sin \pi N^{-1/3}}{\sin(\pi/N)} \leqslant N^{2/3}-1.$$

It follows then from (11) that

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 4-2N^{-2/3}.$$

In the third case we consider $\frac{1}{4} < D_N^* \leqslant \frac{1}{2}$ and $N \geqslant 6$. Note that the additional condition $K > (4N)^{-1}$ for odd N in (10) is then automatically satisfied. From (10) we get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 2+\frac{1}{ND_N^*\sin(\pi/N)} \leqslant 2+\frac{4}{N\sin(\pi/N)}$$

$$\leqslant 2+\frac{4}{6\sin(\pi/6)} \leqslant 4-2\cdot 6^{-2/3} \leqslant 4-2N^{-2/3}.$$

In the fourth case we consider $\frac{1}{4} < D_N^* \leqslant \frac{1}{2}$ and $N < 6$. If $N = 3$, then

$$K = D_N^* - \tfrac{1}{6} \leqslant \tfrac{2}{3} D_N^*,$$

and so from (3), (4), (8), and (9) we get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant \frac{8\pi}{9} \leqslant 4 - 2\cdot 3^{-2/3} = 4 - 2N^{-2/3}.$$

For $N = 4$ we note that $D_N^* \leqslant \frac{1}{2}\cdot 4^{-1/3}$ was already treated in the first case, so that we can assume $D_N^* > \frac{1}{2}\cdot 4^{-1/3}$. From (10) we get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 2 + \frac{1}{ND_N^*\sin(\pi/N)} \leqslant 2 + \frac{4^{1/3}}{2\sin(\pi/4)}$$
$$\leqslant 4 - 2\cdot 4^{-2/3} = 4 - 2N^{-2/3}.$$

For $N = 5$ we use the same argument as for $N = 4$.

In the last case we consider $D_N^* > \frac{1}{2}$. Using the trivial bound for the exponential sum, we get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant \frac{1}{D_N^*} \leqslant 2 \leqslant 4 - 2N^{-2/3}.$$

Thus in all cases we have

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 4 - 2N^{-2/3},$$

and so (2) yields the desired upper bound for c_N.

4. An alternative method. We present an alternative method of establishing an upper bound for c_N that is based on the following variant of a well-known inequality of Koksma (see [1], p. 143, for Koksma's inequality):

LEMMA. *If f is a continuously differentiable function on* $[0, 1]$ *and* $x_1, \ldots, x_N$ *are numbers in* $[0, 1)$ *with discrepancy* D_N^*, *then*

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(t)\,dt\right| \leqslant \left(D_N^* - \frac{1}{2N}\right)\int_0^1 |f'(t)|\,dt + \frac{1}{\sqrt{12}\,N}\Big(\int_0^1 (f'(t))^2\,dt\Big)^{1/2}.$$

Proof. We can assume that $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N$. Put $x_0 = 0$, $x_{N+1} = 1$, and

$$g_N(t) = t - n/N \quad \text{for } x_n \leqslant t < x_{n+1},\ n = 0, 1, \ldots, N,$$

also $g_N(1) = 0$. We note that

$$D_N^* = \sup_{0 \leqslant t \leqslant 1} |g_N(t)|.$$

By [1], p. 143, Lemma 5.1, we have

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(t)\,dt = \int_0^1 g_N(t) f'(t)\,dt. \tag{12}$$

Write again $K = D_N^* - 1/(2N)$ and put

$$h_N(t) = \max(0, |g_N(t)| - K) \quad \text{for } 0 \leqslant t \leqslant 1.$$

Then

$$\begin{aligned}
\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_0^1 f(t)\,dt\right| &\leqslant \int_0^1 |g_N(t)|\,|f'(t)|\,dt \\
&= K\int_0^1 |f'(t)|\,dt + \int_0^1 (|g_N(t)| - K)|f'(t)|\,dt \\
&\leqslant K\int_0^1 |f'(t)|\,dt + \int_0^1 h_N(t)\,|f'(t)|\,dt \\
&\leqslant K\int_0^1 |f'(t)|\,dt + \Big(\int_0^1 h_N^2(t)\,dt\Big)^{1/2}\Big(\int_0^1 (f'(t))^2\,dt\Big)^{1/2}.
\end{aligned}$$

The set $H = \{t \in [0, 1]: h_N(t) > 0\}$ can be divided into at most $2N$ intervals on each of which $h_N(t)$ is a linear function of slope ± 1. Hence

$$\int_0^1 h_N^2(t)\,dt = \int_H h_N^2(t)\,dt \leqslant 2N \int_0^{1/(2N)} t^2\,dt = \frac{1}{12N^2},$$

and the Lemma is established.

COROLLARY. *For any real numbers $x_1, \ldots, x_N$ with discrepancy D_N^* we have*

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| \leqslant 4D_N^* - \left(2 - \frac{\pi}{\sqrt{6}}\right)\frac{1}{N}.$$

Proof. We can assume that $0 \leqslant x_n < 1$ for $1 \leqslant n \leqslant N$. For a suitable real θ we have

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| = \frac{1}{N}\sum_{n=1}^{N} e(x_n - \theta),$$

and taking real parts we get

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| = \frac{1}{N}\sum_{n=1}^{N} \cos 2\pi(x_n - \theta). \tag{13}$$

Applying the Lemma with $f(t) = \cos 2\pi(t-\theta)$, we obtain

$$\left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| \leqslant \left(D_N^* - \frac{1}{2N}\right)\int_0^1 |f'(t)|\,dt + \frac{1}{\sqrt{12}\,N}\left(\int_0^1 (f'(t))^2\,dt\right)^{1/2}$$
$$= 4D_N^* - \left(2 - \frac{\pi}{\sqrt{6}}\right)\frac{1}{N}.$$

We establish now an upper bound for c_N. Let $a > 0$ be a real number to be determined later and consider first the case where $D_N^* \leqslant aN^{-1/3}$. Then, by the Corollary,

$$\text{(14)}\qquad \frac{1}{ND_N^*}\left|\sum_{n=1}^{N} e(x_n)\right| \leqslant 4 - \left(2 - \frac{\pi}{\sqrt{6}}\right)\frac{1}{ND_N^*} \leqslant 4 - \left(2 - \frac{\pi}{\sqrt{6}}\right)a^{-1}N^{-2/3}.$$

Next we consider the case where $aN^{-1/3} < D_N^* \leqslant \frac{1}{3}$. By (12) and (13) we get, with $D = D_N^*$ and $f(t) = \cos 2\pi(t-\theta)$ for a suitable θ,

$$\text{(15)}\qquad \left|\frac{1}{N}\sum_{n=1}^{N} e(x_n)\right| = \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_0^1 g_N(t) f'(t)\,dt$$
$$\leqslant \int_0^1 |g_N(t)|\,|f'(t)|\,dt = D\int_0^1 |f'(t)|\,dt + \int_0^1 (|g_N(t)| - D)\,|f'(t)|\,dt$$
$$= 4D - 2\pi\int_0^1 (D - |g_N(t)|)\,|\sin 2\pi(t-\theta)|\,dt.$$

We note that there exists an interval $[t_1, t_2] \subseteq [0, 1]$ such that $t_2 - t_1 = D$ and either

$$D - |g_N(t)| \geqslant t - t_1 \quad \text{for all } t \in (t_1, t_2)$$

or

$$D - |g_N(t)| \geqslant D + t_1 - t \quad \text{for all } t \in (t_1, t_2).$$

With a suitable change of variable we obtain

$$\text{(16)}\qquad \int_0^1 (D - |g_N(t)|)\,|\sin 2\pi(t-\theta)|\,dt \geqslant \int_0^D t\,|\sin 2\pi(t-\beta)|\,dt$$

for some real β. With $b = 1 + \sqrt{3}$ we get

$$\int_0^D t\,|\sin 2\pi(t-\beta)|\,dt \geqslant D\sum_{k=1}^{\infty} b^{-k}\int_{b^{-k}D}^{b^{-k+1}D} |\sin 2\pi(t-\beta)|\,dt$$
$$\geqslant 2D\sum_{k=1}^{\infty} b^{-k}\int_0^{\frac{1}{2}(b-1)b^{-k}D} \sin 2\pi t\,dt$$
$$= \frac{D}{\pi}\sum_{k=1}^{\infty} b^{-k}(1 - \cos\pi(b-1)b^{-k}D).$$

Since $D \leqslant \frac{1}{3}$, we have $\pi(b-1)b^{-k}D \leqslant \frac{2}{3}$, and so we can use

$$1-\cos x \geqslant \frac{x^2}{2}\left(1-\frac{x^2}{12}\right) \geqslant \frac{13}{27}x^2 \quad \text{for } 0 \leqslant x \leqslant \frac{2}{3}.$$

This yields

$$\int_0^D t\,|\sin 2\pi(t-\beta)|\,dt \geqslant \frac{13\pi}{27}(b-1)^2 D^3 \sum_{k=1}^{\infty} b^{-3k} = \frac{13\pi}{27(2\sqrt{3}+3)} D^3.$$

Together with (15) and (16) we get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 4 - \frac{26\pi^2}{27(2\sqrt{3}+3)}(D_N^*)^2 \leqslant 4-(1.47)a^2 N^{-2/3}.$$

Equating this bound with the one in (14), we find the optimal choice

$$a = \left(\frac{2-(\pi/\sqrt{6})}{1.47}\right)^{1/3}.$$

This yields

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant 4-(0.9)N^{-2/3}$$

whenever $D_N^* \leqslant \frac{1}{3}$. For $D_N^* > \frac{1}{3}$ we use the trivial bound for the exponential sum to get

$$\frac{1}{ND_N^*}\Big|\sum_{n=1}^{N} e(x_n)\Big| \leqslant \frac{1}{D_N^*} \leqslant 3 \leqslant 4-(0.9)N^{-2/3},$$

and so by (2) we have the upper bound

$$c_N \leqslant 4-(0.9)N^{-2/3}.$$

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