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A LOCAL-NON-GLOBAL HUREWICZ FIBRATION

BY

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1. Introduction. Our aim is to describe an example of a map $p: X \to L$ which is not a Hurewicz fibration although it is a *local Hurewicz fibration*, i.e., there is an open cover $\mathscr{C}$ of L such that, for each $U \in \mathscr{C}$, $(p^{-1}(U), p|p^{-1}(U), U)$ is a Hurewicz fibration.

The well-known uniformization theorem of Hurewicz [3] states: *A local fibration over a paracompact space is a fibration.*

Variations and generalizations in [1] deal with maps whose restrictions to the elements of a numerable cover of the base are fibrations. In the proofs of each such local-to-global result, the local finiteness condition (paracompactness or numerable cover) seems necessary. Our example shows that this is the case.

2. The example (X, p, L).

2.1. Definitions and notation. We let L denote the long line ([5], p. 71) described briefly as follows. Ω is the set of all ordinals less than the first uncountable ordinal well ordered by $<$. With I, the unit interval, $L = \Omega \times I/\sim$, where the only identifications are $(a, 1) \sim (a+1, 0)$ for each $a \in \Omega$. For $0 \leqslant t < 1$ we write (a, t) for the equivalence class of (a, t) and note that each element of L can be written uniquely in this way. The space L has the quotient topology. It is also an ordered space with the obvious order. By identification of a with $(a, 0)$ we consider Ω as a subspace of L. The uncountability and sequential compactness of Ω (and hence L) will be the main properties used in our proofs.

Next, we define X as a subset of $L \times I$, and p: $X \to L$ as the first coordinate projection. To do this, let f: $I \to I$ be defined by $f(x) = 2x$ for $0 < x \leqslant \frac{1}{2}$ and by $f(x) = 2-2x$ for $\frac{1}{2} \leqslant x \leqslant 1$, and set

$$X = \big(\{((a, t), s) \in L \times I \,|\, 0 < s \leqslant f(t)\}\big) \cup (\Omega \times \{0\}).$$

In other words, we obtain X by attaching solid triangles to each "interval" of L along one edge and then removing the "interiors" of the attached edges. Finally, we set $p((a, t), s) = (a, t)$ and observe that

(1) $p^{-1}(a, 0) = \{(a, 0), 0\}$ is a single point,

(2) for $0 < t < 1$, $((a, t), s) \in X$ implies $s > 0$.

2.2. *(X, p, L) is not a fibration.* Recall that (X, p, L) is a *Hurewicz fibration* if and only if there is a continuous lifting function

$$\lambda\colon P \to X^I, \qquad \text{where } P = \{(x, \omega) \in X \times L^I \mid p(x) = \omega(0)\},$$

such that

$$p\big(\lambda(x, \omega)(t)\big) = \omega(t) \quad \text{and} \quad \lambda(x, \omega)(0) = 0$$

(see [3]). We assume such a λ and reach a contradiction by a standard uncountability argument.

We let $\omega_a \in L^I$ be defined by $\omega_a(t) = (a, t)$. It follows from observation (2) in 2.1 and the uncountability of Ω that there are an uncountable set $A \subset \Omega$ and an $s_0 > 0$ such that

$$\lambda\big((a, 0), \omega_a\big)(\tfrac{1}{2}) = \big((a, \tfrac{1}{2}), s\big) \quad \text{with } s > s_0 \text{ for each } a \in A.$$

By the sequential compactness of Ω, there are a sequence (a_i) in A and a limit ordinal $a_0 \in \Omega$ with $(a_i) \to a_0$ and $a_i < a_0$ for each i. The following statements are now easy to check:

1. $\big((a_i, 0), \omega_{a_i}\big) \to \big((a_0, 0), \tilde{a}_0\big)$ in P, where $\tilde{a}_0$ denotes the constant path at a_0.

2. $\pi_2\big(\lambda\big((a_i, 0), \omega_{a_i}\big)(\tfrac{1}{2})\big) > s_0$, where π_2 denotes the second coordinate projection.

3. $\pi_2\big(\lambda\big((a_0, 0), \tilde{a}_0\big)(\tfrac{1}{2})\big) = 0$ by observation (1) of 2.1.

4. $\big(\lambda\big((a_i, 0), \omega_{a_i}\big)\big) \not\to \lambda\big((a_0, 0), \tilde{a}_0\big)$ in X^I.

Statements 1 and 4 contradict the continuity of λ and our proof is complete.

2.3. *(X, p, L) is a local fibration.* We shall show that (X, p, L) has the slicing structure property [2], [6]. From this the local fibration condition follows easily. Specifically, for each $a_0 \in \Omega$ we let $B = \{x \in L \mid x < a_0\}$, where $x < a_0$ means $x = (a, t)$ with $a < a_0$, and let $E = p^{-1}(B)$. We define a (continuous) slicing function $\gamma\colon E \times B \to E$ by

$$\gamma\big(e, p(e)\big) = e \quad \text{and} \quad p\gamma(e, b) = b \quad \text{for each } (e, b).$$

The countability of $B^* = \{a \in \Omega \mid a < a_0\}$ will be crucial in defining γ.

Step 1. Let $B^* = \{a_1, a_2, a_3, \ldots\}$ and define $k\colon B \to I$ by

$$k(a_n, r) = \min\left\{\frac{1}{n}, f(r)\right\},$$

where we note $0 \leqslant r < 1$. Since $(r_i) \to 0$ and $(r_i) \to 1$ both imply that $(f(r_i)) \to 0$, it is easy to check the continuity of k at any point except possibly at $(a_n, 0)$, where a_n is a cluster point for B^*. In such a case, the countability of B^* implies that, for each $\varepsilon > 0$, there is an open set U about a_n in B^* such that $1/m < \varepsilon$ whenever $a_m \in U$. The continuity at $(a_n, 0)$ follows easily.

Since $k(a_n, r) \leqslant f(r)$, we could use k to define a section from B to E. Functions obtained from k will yield the collection of sections required for the slicing function γ.

Step 2. For each natural number n we define $g_n\colon S \times I \to I$, where $S \subset I \times I$ and $S = \{(t, s) \mid s \leqslant f(t)\}$. For $x = ((t, s), r) \in S \times I$ the definition is as follows:

I. for $0 \leqslant t \leqslant 1/2n$ and $0 \leqslant r \leqslant t$,

$$g_n(x) = \min\{f(r), s\};$$

II. for $0 \leqslant t \leqslant 1/2n$ and $t \leqslant r \leqslant 1/2n$,

$$g_n(x) = f(r) - 2t + s;$$

III. for $0 \leqslant t \leqslant 1/2n$ and $1/2n \leqslant r \leqslant 1/2$,

$$g_n(x) = \frac{1}{n} - 2t + s;$$

IV. for $0 \leqslant t \leqslant 1/2n$ and $1/2 \leqslant r < 1$,

$$g_n(x) = g_n((t, s), 1 - r);$$

V. for $1/2n \leqslant t \leqslant 1/2$,

$$g_n(x) = \min\{f(r), s\};$$

VI. for $1/2 \leqslant t < 1$,

$$g_n(x) = g_n((1 - t, s), r).$$

Fig. 1 illustrates the g_n. Each such function is piecewise-linear in r with seven "pieces" and critical points continuously dependent on t and s. Continuity of each g_n follows easily. These functions have the following properties:

$$g_n((t, s), t) = s, \tag{1}$$

$$g_n((t, s), r) \leqslant f(r), \tag{2}$$

$$g_n((0, 0), r) = g_n((1, 0), r) = \min\left\{f(r), \frac{1}{n}\right\}, \tag{3}$$

$$g_n((t, s), r) \leqslant \max\{s, k(a_n, r)\}. \tag{4}$$

Step 3. Now we define $\varphi\colon E \times B \to I$ by setting

$$x = \big(((a_i, t), s), (a_n, r)\big)$$

and putting

$$\varphi(x) = \begin{cases} k(a_n, r) & \text{if } a_i \neq a_n, \\ g_n\big((t, s), r\big) & \text{if } a_i = a_n. \end{cases}$$

Here we suppose $0 \leqslant s < 1$ and $0 \leqslant r < 1$. The continuity at any point x with $s \neq 0$ and $r \neq 0$ follows immediately from the continuity of k and that of the g_n. If $s = 0$, $r \neq 0$ and $a_n = a_i$, the continuity at x follows from the continuity of g_n. If $s = 0$, $r \neq 0$, and $a_i \neq a_n$, the conti-

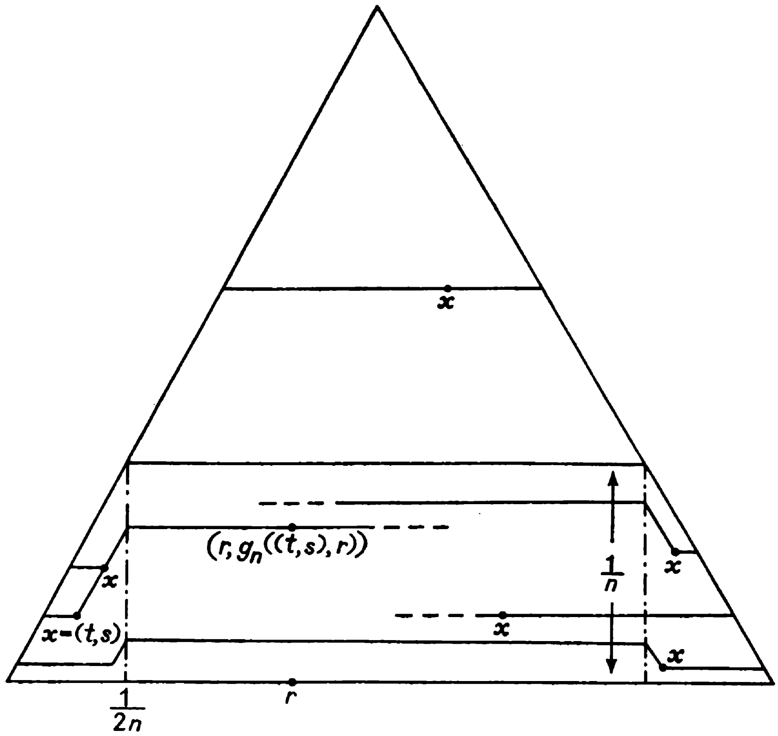

Fig. 1. The function g_n: $S \times I \to S$ for various "fixed" $x = (t, s) \in S$ and variable $r \in I$

nuity at x follows from the continuity of k and, in case where a_i is the successor of a_n, the fact that $t_i \to 1$ implies

$$\big\{g_n\big((t_i, 0), r\big)\big\} \to k(a_n, r).$$

If $s = 0$ and $r = 0$, then $\varphi(x) = 0$ and the continuity follows from the continuity of k and properties of the g_n. In case where a_n is a cluster point for A^*, the property

$$g_n\big((t, s), r\big) \leqslant \max\{s, k(a_n, r)\}$$

is needed. Note that $\varphi\Big(\big(((a_i, t), s\big), (a_i, t)\Big) = s$.

Step 4. Now we complete our proof by defining γ: $E \times B \to E$ by $\gamma(e, b) = \big(b, \varphi(e, b)\big)$. The continuity is immediate and we have $q\big(\gamma(e, b)\big) = b$. Furthermore,

$$\gamma\big(e, q(e)\big) = \gamma\Big(\big((a_i, t), s\big), (a_i, t)\Big) = \Big((a_i, t), \varphi\Big(\big((a_i, t), s\big), (a_i, t)\Big)\Big)$$
$$= \big((a_i; t), s\big) = e.$$

3. Remarks. If we replaced $A \times \{0\}$ by $L \times \{0\}$ in the definition of X, i.e., we added the "interiors" of the base lines of the triangles, then we would obtain a triple (X^*, p, L) which would be a Hurewicz fibration. Thus, the omission of the "interiors" is crucial.

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