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A NOTE ON PERMANENTLY SINGULAR ELEMENTS IN TOPOLOGICAL ALGEBRAS

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1. Introduction. In [1] Arens showed that in a Banach algebra X, if c is not a topological divisor of zero (t.d.z.), i.e.

$$\inf_{x\neq 0} \|cx\|/\|x\| > 0,$$

then X can be embedded isomorphically and isometrically in an algebra Y in which c has an inverse. From this it follows that the permanently singular (not invertible in any superalgebra) elements of an algebra X are the singular elements which are t.d.z.'s. In [5] Michael considered two t.d.z. concepts for locally multiplicatively convex Hausdorff (LMCH) algebras: t.d.z.'s and strong t.d.z.'s. In [7] it is shown that the permanently singular elements of an LMCH algebra are the t.d.z.'s, and in [4] it is proved that these two concepts of topological divisors of zero are not the same even in Fréchet Q-algebras.

There are essentially three complete valued fields carrying rank one valuation: the real numbers, complex numbers, and non-archimedean valued fields. The non-archimedean valued fields form a large category of fields [2]. In this paper we show that the above-quoted results are true in non-archimedean locally multiplicatively convex Hausdorff (NLMCH) algebras over complete rank one non-archimedean valued fields. This represents an extension of the previous theory.

In [1] it is shown that if x is in the boundary of the singular elements of a Banach algebra X, then x is a t.d.z. This result is proved here for non-archimedean Banach algebras. If X is a Banach algebra over the complex numbers, this result together with the Gelfand theory and connectedness of the complex plane yield the well-known theorem of Shilov that the only complex Banach algebra with no t.d.z.'s other than 0 is the complex numbers. We show here that this result is not true for non--archimedean Banach algebras.

Throughout this paper X denotes a commutative algebra with identity e over a complete rank one non-trivially non-archimedean valued field F.

2. Inverse producing extensions of topological algebras. In this section we consider inverse producing extensions of topological algebras. We characterize those elements of an NLMCH algebra X for which a superalgebra Y of X can be found in which they become invertible or, alternatively, those elements which are permanently singular in X. We begin with the Banach algebra problem.

THEOREM 1. *Let X be a non-archimedean Banach algebra over F and let $c \in X$ be singular and not a topological divisor of zero. Then there exists a superalgebra Y of X in which c is invertible.*

Proof. We use ideas similar to those of Arens [1] but with modifications necessary for his procedure to fit the non-archimedean setting. Let $X[[z]]$ denote the algebra of all formal power series in the transcendental element z with coefficients from X. As c is not a t.d.z., choose a real number $t > 0$ such that $\|cx\| \geqslant t^{-1}\|x\|$ for all $x \in X$. Let

$$X[[z, t]] = \left\{\sum x_n z^n \colon \|x_n\| t^n \to 0\right\}.$$

On $X[[z, t]]$ we define $\left\|\sum x_n z^n\right\| = \max \|x_n\| t^n$. As X is a non-archimedean Banach algebra, it is readily shown that so is $X[[z, t]]$. Since all elements of the verification of this fact are similar, we present only the proof that $X[[z, t]]$ is multiplicatively closed. Let $f = \sum x_n z^n$ and $g = \sum y_n z^n$. Then $fg = \sum w_n z^n$, where $w_n = \sum x_j y_{n-j}$. We choose n large enough so that in the sum for w_n either j or $n-j$ is such that

$$\|x_j\| \leqslant t^{-j}\varepsilon/\max\{\|f\|, \|g\|\} \quad \text{or} \quad \|y_{n-j}\| \leqslant t^{n-j}\varepsilon/\max\{\|f\|, \|g\|\}.$$

Then, by the assumption that X is a non-archimedean Banach algebra, the strong triangle inequality can be applied to produce $\|w_n\| t^n \leqslant \varepsilon$. Thus $\|w_n\| t^n \to 0$ and $fg \in X[[z, t]]$.

Now let J be the smallest closed ideal generated by $1-cz$. We will show (Proposition 2) that in this setting the principal ideal generated by $1-cz$ is always equal to J. Then $X[[z, t]]/J = Y$ is a non-archimedean Banach algebra and $c+J$ has an inverse $z+J$ in Y. As in the case Arens considered [1], the problem of significance is that of showing that X is isometrically embedded in Y. Surely, if $x \in X$, then $\|x+J\| \leqslant \|x\|$. We wish to show that, for any power series $f \in X[[z, t]]$, it follows that

$$\|x-(e-cz)f\| \geqslant \|x\|$$

and the proof will be done. Clearly, if $x = 0$, this inequality holds and we can assume that $x \neq 0$. Writing $f = \sum x_n z^n$, we have

$$x-(e-cz)f = (x-x_0)+(cx_0-x_1)z+\sum_{n=2}^{\infty}(cx_{n-1}-x_n)z^n.$$

Suppose that $\|x_0\| \neq \|x\|$. As the norm on X satisfies the strong triangle inequality, it follows that

$$\|x-x_0\| \geqslant \|x\| \quad \text{and} \quad \|x-(e-cz)f\| \geqslant \|x-x_0\| \geqslant \|x\|.$$

Now suppose that $\|x_0\| = \|x\|$. Under this assumption, if $\|cx_0\| \neq \|x_1\|$, then

$$\|cx_0-x_1\|t \geqslant \|cx_0\|t \geqslant \|x_0\|t^{-1}t = \|x_0\| = \|x\|$$

and once again the proof is done. We now observe that if we repeat the assumptions that $\|x_0\| = \|x\|$ and $\|cx_{n-1}\| = \|x_n\|$ for all n, we contradict a property of f. For then

$$\|x_n\| = \|cx_{n-1}\| \geqslant \|x_{n-1}\|t^{-1} = \|cx_{n-2}\|t$$

and, ultimately, we obtain the inequality $\|x_n\| \geqslant \|x_0\|t^{-n}$. But then $\|x_n\|t^n \geqslant \|x_0\| = \|x\|$ which contradicts $\|x_n\|t^n \to 0$. Thus $\|cx_{n-1}\|$ cannot always be equal to $\|x_n\|$, and if we let n_0 be the first index at which $\|cx_{n_0-1}\| \neq \|x_{n_0}\|$, we have

$$\begin{aligned}\|x-(e-cz)f\| &\geqslant \|cx_{n_0-1}-x_{n_0}\|t^{n_0} \geqslant \|cx_{n_0-1}\|t^{n_0} \geqslant \|x_{n_0-1}\|t^{n_0-1}\\ &= \|cx_{n_0-2}\|t^{n_0-1} \geqslant \|x_{n_0-2}\|t^{n_0-2} \geqslant \ldots \geqslant \|x_0\| = \|x\|\end{aligned}$$

and the proof is complete.

We can now refer to the proofs of [4], [5] and [7] and state that the following is true:

THEOREM 2. *Let X be an NLMCH algebra. Then $c \in X$ is singular in every superalgebra Y of X if and only if there exists a base of non-archimedean seminorms $\{p_a\}$ for the topology of X such that $\pi_a c = c+p_a^{-1}(0)$ is permanently singular (a t.d.z.) in each factor algebra X_a, X_a denoting the completion of $X/p_a^{-1}(0)$.*

In an NLMCH algebra X, a *strong t.d.z.* is an element x such that, for some net (x_β) from X with $x_\beta \not\to 0$, $xx_\beta \to 0$. A topological divisor of zero is an element $x \in X$ such that, for any base $\{p_a\}$, $\pi_a x$ is a t.d.z. in some factor algebra X_a. As in the classical case, we can now say that the following are equivalent in any NLMCH algebra X:

(1) $x \in X$ is a t.d.z.

(2) There exists a base $\{p_a\}$ for the topology of X of non-archimedean seminorms such that $\pi_a x$ is a t.d.z. in each factor algebra X_a.

(3) x is permanently singular in X.

For the proof of these statements, see [7].

Michael [5] raised the question of whether or not the two notions of a t.d.z. are the same in LMCH algebras. They are, indeed, the same in Banach algebras. Kuczma [4] answered this question in the negative with an example. All of the essential properties of this example go through if one assumes the underlying field to be non-archimedean valued and makes a few changes in the algebra so that it fits the setting of the present paper.

Example 1. Let X be the algebra of all sequences (a_m) with $a_m \in F$ and Cauchy product as the multiplication in X. Let a topology on the algebra X be generated by the sequence of seminorms p_n, where $p_n\big((a_m)\big) = \sup\{|a_0|, \ldots, |a_n|\}$. Then the following are true:

(1) X is an NLMCH Fréchet Q-algebra.

(2) $M = \{(a_m) \mid a_0 = 0\}$ is the unique maximal ideal of X and is the kernel of the homomorphism $h: X \to F$ such that $h\big((a_m)\big) = a_0$.

(3) If (a_m) is a singular element in X, then $\pi_n(a_m) = (0, \ldots, a_n)$ is the limit of a sequence of invertible elements in X_n and is, therefore, a t.d.z. (see Theorem 3). Thus (a_m) is a t.d.z. in X.

(4) There are no strong t.d.z.'s in X.

3. Some properties of $X[[z, t]]/J$. A *Gelfand algebra* is an NLMCH algebra such that every closed maximal ideal is the kernel of a non-trivial continuous homomorphism of X into the underlying field F. Let X be a non-archimedean Banach algebra. Thus all maximal ideals are closed in X. Then X is a Gelfand algebra if and only if every maximal ideal is the kernel of such a homomorphism. As every non-archimedean valued field admits a proper extension field K which is non-archimedean valued, there is a Banach algebra $X = K$ over every such F which is not a Gelfand algebra. Of course, the complex numbers admit no such possibility.

Let us assume that the algebra X of Theorem 1 is a Gelfand algebra. If we could prove that the inverse producting extension $Y = X[[z, t]]/J$ of X is a Gelfand algebra, then it could be shown (as in [3]) that if the factor algebras of X are Gelfand algebras, then X is a Gelfand algebra — and thus it would immediately follow that the assertion of Theorems 1 and 2 are valid with all algebras in these theorems being Gelfand algebras. For example, $c \in X$ is singular in every Gelfand superalgebra of a Gelfand algebra X if and only if c is a t.d.z.

As the question of whether $Y = X[[z, t]]/J$ is a Gelfand algebra remains open, we present two results concerned with properties of Y which might be of value in ultimately answering this question. Proposition 1 is stronger than the analogous result in the classical situation, and Proposition 2 has no classical analogue.

PROPOSITION 1. (a) *The principal ideal generated by* $e-cz$ *in* $X[[z,t]]$ *is closed.*

(b) *The mapping* $T\colon X[[z,t]]\to X[[z,t]]$ *is continuous,* $f\to(e-cz)f$, *and has a continuous inverse* T^{-1} *with* $\|T^{-1}\|\leqslant 1$.

Proof. Both (a) and (b) would be proved if it is shown that $\|(e-cz)f\|\geqslant\|f\|$ for all $f\in X[[z,t]]$. We consider a number of cases. Let n_0 be such that $\|f\|=\|x_{n_0}\|t^{n_0}$, where $f=\sum x_n z^n$. If $n_0=0$, then, as

$$\|(e-cz)f\|=\sup\{\|x_0\|,\ \|cx_0-x_1\|t,\ldots,\ \|cx_{n-1}-x_n\|t^n,\ldots\}\geqslant\|x_0\|=\|f\|,$$

we are done in this case.

Hence we can assume that $n_0>0$.

(1) Let us suppose that $\|x_{n_0}\|\neq\|cx_{n_0-1}\|$. Then

$$\|(e-cz)f\|\geqslant\|cx_{n_0-1}-x_{n_0}\|t^{n_0}\geqslant\|x_{n_0}\|t^{n_0}=\|f\|$$

and we are done again.

(2) Let $\|x_{n_0}\|=\|cx_{n_0-1}\|$ and $\|x_{n_0+1}\|>\|cx_{n_0}\|$. Then

$$\|x_{n_0+1}\|t^{n_0+1}>\|cx_{n_0}\|t^{n_0+1}\geqslant\|x_{n_0}\|t^{-1}t^{n_0+1}\geqslant\|x_{n_0}\|t^{n_0}=\|f\|$$

which is a contradiction.

(3) Let us assume that $\|cx_{n_0-1}\|=\|x_{n_0}\|$ and $\|x_{n_0+1}\|<\|cx_{n_0}\|$. Then

$$\|cx_{n_0}-x_{n_0+1}\|t^{n_0+1}=\|cx_{n_0}\|t^{n_0+1}\geqslant\|x_{n_0}\|t^{n_0}=\|f\|.$$

(4) Let $\|cx_{n_0-1}\|=\|x_{n_0}\|$ and $\|cx_{n_0}\|=\|x_{n_0+1}\|$.

(a) If $\|cx_{n_0}\|>\|x_{n_0}\|t^{-1}$, then

$$\|x_{n_0+1}\|>t^{-1}\|x_{n_0}\|\quad\text{and}\quad\|x_{n_0+1}\|t^{n_0+1}>\|x_{n_0}\|t^{n_0}=\|f\|$$

which is a contradiction.

(b) If $\|cx_{n_0}\|=\|x_{n_0}\|t^{-1}$, then

$$\|x_{n_0+1}\|t^{n_0+1}=\|x_{n_0}\|t^{n_0}=\|f\|.$$

If we come to case (4b) through the entire cycle of possibilities, then we start the cycle of possibilities over again, but we replace n_0 by n_0+1 in all arguments. This is permissible as (4b) produces the relationship $\|f\|=\|x_{n_0+1}\|t^{n_0+1}$. If we do not arrive at (4b), then we are done. However, we cannot continue to return to (4b) indefinitely, for then $\|f\|=\|x_n\|t^n$ indefinitely and this violates the condition $\|x_n\|t^n\to 0$. Hence the proof is complete.

PROPOSITION 2. *With* X *as in Theorem* 1 *and* $Y=X[[z,t]]/J$, *it follows that* $\|z^n+J\|=t^n$ *for all positive integers* n.

Proof. We begin with the case $n=1$. Since

$$\begin{aligned}\|z+J\| &= \inf\{\|z-(e-cz)f\|:\ f\in X[[z,t]]\}\\ &= \inf\Big\{\|-x_0+(e+cx_0-x_1)z+\sum(cx_{n-1}-x_n)z^n\|:\ \sum x_nz^n\in X[[z,t]]\Big\}\\ &\leqslant \|z\|=t,\end{aligned}$$

we wish to show that in no case is it possible that

$$(*)\qquad \Big\|-x_0+(e+cx_0-x_1)z+\sum(cx_{n-1}-x_n)z^n\Big\|<t.$$

If $(*)$ occurs, then both x_0 and x_1 cannot be zero. In addition, the following relations must all hold:

(1) $\|x_0\|t^{-1}<1$;

(2) $\|e+cx_0-x_1\|<1$;

(3) $\|cx_{n-1}-x_n\|t^{n-1}<1,\ n\geqslant 2$.

Inequality (2) can occur in four ways:

(a) $\|cx_0\|>1,\ \|x_1\|>1,\ \|x_1\|=\|cx_0\|$;

(b) $\|cx_0\|=1,\ \|x_1\|=1$;

(c) $\|cx_0\|=1,\ \|x_1\|<1$;

(d) $\|cx_0\|<1,\ \|x_1\|=1$.

Examining these cases one at a time, we find that, assuming case (a) occurs, we have $\|cx_1\|\geqslant\|x_1\|t^{-1}>t^{-1}$ and, therefore, $\|cx_1-x_2\|t<1$ can occur only if $\|x_2\|=\|cx_1\|$. Hence it follows that

$$\|x_2\|=\|cx_1\|\geqslant\|x_1\|t^{-1}>t^{-1}.$$

Similarly, $\|cx_2-x_3\|t^2<1$ can occur only if $\|cx_2\|=\|x_3\|$, since

$$\|cx_2\|\geqslant t^{-1}\|x_2\|=t^{-1}\|cx_1\|>t^{-2}.$$

Hence

$$\|x_3\|\geqslant t^{-1}\|x_2\|\geqslant t^{-1}\|cx_1\|\geqslant t^{-2}\|x_1\|>t^{-2}.$$

Proceeding in this way to satisfy $(*)$, we continue to require that $\|cx_{n-1}\|=\|x_n\|$ and, therefore, ultimately, that $\|x_n\|>t^{-(n-1)}$, which is a contradiction as $\|x_n\|t^n\to 0$.

Assuming case (b) occurs, we are led to a contradiction exactly as in (a) except for the minor difference that, for all n, the contradiction $\|x_n\|\geqslant t^{-(n-1)}$ is obtained.

The assumption of case (c) leads to the contradiction that $\|e+cx_0\|<1$ from which cx_0 and, therefore, c is invertible. It goes without saying that we are only considering the case where c is not invertible.

If we assume that (d) occurs, we proceed to a contradiction, as in cases (a) and (b), with $\|x_n\| \geqslant t^{-(n-1)}$.

To prove that $z^2-(e-cz)\sum x_n z^n$ always has the norm greater than or equal to t^2 we must, as in the first case, use the following relationships:

(1) $\|x_0\|t^{-2} < 1$;

(2) $\|cx_0-x_1\|t^{-1} < 1$;

(3) $\|e+cx_1-x_2\| < 1$;

(4) $\|cx_{n-1}-x_n\|t^{n-2} < 1,\ n \geqslant 3$.

We begin the argument at inequality (3) and proceed to contradictions exactly as in the previous case. The case for $z^m-(e-cz)\sum x_n z^n$ (arbitrary m) follows in the same way.

4. Boundary of the singular elements. If X is a complex Banach algebra and if $x \in X$ is in the boundary of the singular elements, then x is a t.d.z. We prove this result here for non-archimedean Banach algebras (Theorem 3). An important application of the classical version of this theorem is the well-known result of Shilov that a Banach algebra without topological divisors of zero (save the zero vector) must be the complex numbers. This follows from the connectedness of the complex numbers and the resulting existence of a complex number in the boundary of the spectrum of each $x \in X$.

It is not difficult, in the non-archimedean setting, to construct examples of algebras in which there will be vectors whose spectrum is clopen (closed and open). Thus a proof of an analogue of Shilov's theorem would have to proceed in an entirely different manner. In fact, Shilov's theorem is not, generally, true in this setting. Indeed, for each F, there is an algebra X over F in which there are no t.d.z.'s and yet $X \neq F$. In Examples 3 and 4 we present two different types of algebras X in which this occurs for each field F. The algebras of Example 4 are never Gelfand algebras while if F is algebraically closed, those of Example 3 are always Gelfand algebras ([6], p. 171, and [8], p. 162). Shilov's theorem has not yet been proved in any significant category of non-archimedean Banach algebras save the Gelfand V^*- algebras (see [6], p. 148, and Example 2).

THEOREM 3. *If X is a non-archimedean Banach algebra and x belongs to the boundary of the singular elements, then x is a t.d.z.*

Proof. Let $x_n \to x$ and let y_n be such that $x_n y_n = e$. Then $y_n x - e = y_n(x-x_n)$. As x cannot have an inverse, it follows that

$$1 \leqslant \|y_n x - e\| \leqslant \|y_n\|\,\|x-x_n\|.$$

Thus it follows that

$$(**)\qquad 1/\|y_n\| \leqslant \|y_n(x-x_n)\|/\|y_n\| \leqslant \|x-x_n\| \to 0.$$

We wish to show that $(**)$ implies that $\|y_n x\|/\|y_n\| \to 0$ and, therefore, that x is a t.d.z. As $\|y_n\|$ cannot be treated as a scalar in this setting, we must consider two cases:

(1) F is discretely valued and $|a| = r < 1$ is a generator of the value group of F.

(2) The value group of F is dense in the non-negative real numbers.

Considering case (1) first, by $(**)$, if k_n is an integer such that $r^{k_n} \leqslant \|y_n\| \leqslant r^{k_n-1}$, then $1/r^{k_n} \geqslant 1/\|y_n\| \geqslant 1/r^{k_n-1}$ and, by $(**)$, all three sequences in this inequality tend towards zero. Thus, as $\|x_n y_n/a^{k_n}\|$ and $\|x_n y_n/a^{k_n-1}\|$ tend towards zero, it follows that $y_n x/a^{k_n}$ and $y_n x/a^{k_n-1}$ tend towards the zero vector. Since

$$\|y_n x/a^{k_n}\| = \|y_n x\|/r^{k_n} \geqslant \|y_n x\|/\|y_n\|,$$

the proof is done in case (1).

In case (2) we can approximate $\|y_n\|$, as closely as we wish, by values of scalars in F and complete the proof in a direct manner.

COROLLARY 1. *If a scalar a is in the boundary of the spectrum of a vector $x \in X$, then $x - ae$ is a t.d.z.*

COROLLARY 2. *If a vector x belongs to the intersection of all maximal ideals of X, then x is a t.d.z.*

Proof. The spectrum of x consists solely of the scalar 0.

Example 2. Let T be a compact Hausdorff zero-dimensional space and $C(T, F)$ the Banach algebra of all continuous functions taking T into F with sup norm. If $f \in C(T, F)$ is singular, then $z(f) = \{t \in T \mid f(t) = 0\}$ is not empty. Let $O_n = \{t \in T \mid |f(t)| < 1/n\}$. Then O_n is clopen and $z(f) = \bigcap O_n$. Let

$$g_n(t) = \begin{cases} a_n, & 0 < |a_n| < 1/n,\ t \in O_n, \\ f(t), & t \in CO_n. \end{cases}$$

Then g_n is invertible for each n and $g_n \to f$. Hence f is a t.d.z.

Now, if X is a V^*-algebra and a Gelfand algebra with no t.d.z. save the zero vector, $X = F$. We refer to Example 2 and [6].

Example 3. Let x be transcendental over F and let

$$X = \left\{ \sum a_n x^n \;\middle|\; a_n \in F \text{ and } |a_n| \to 0 \right\}.$$

Let $\left\| \sum a_n x^n \right\| = \max |a_n|$. Then X is a non-archimedean Banach algebra over F. Since the norm on X is multiplicative, there are no t.d.z.'s in X. If F is algebraically closed, the maximal ideals of X are either principal ideals generated by $x-a$ with $|a| \leqslant 1$ or ideals which contain no non-zero polynomials in x. If Shilov's theorem were true for Gelfand algebras, then X would not be a Gelfand algebra. However, by [8], p. 162, X is a Gelfand algebra. Thus Shilov's theorem fails to be true for Gelfand algebras.

Example 4. For every complete non-archimedean valued field F, there exists a proper rank one non-archimedean valued extension field X.

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