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THE ORDER OF NORMALITY AND MEROMORPHIC UNIVALENT FUNCTIONS

BY

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1. Introduction. The *order of normality* $\omega(f)$ of a function f meromorphic in $D = \{|z| < 1\}$ is defined by

$$\omega(f) = \sup_{z \in D}(1-|z|^2)f^{\#}(z), \qquad \text{where } f^{\#} = |f'|/(1+|f|^2)$$

(see [5]); f is *normal* in D if and only if $\omega(f) < +\infty$ ([2], Theorem 3). Let $\mathcal{F}$ be a family of meromorphic functions in D, and let

$$\omega(\mathcal{F}) = \sup_{f \in \mathcal{F}} \omega(f);$$

thus, it is clear what is meant by $\omega(\mathcal{F}')$, where

$$\mathcal{F}' = \{f'; f \in \mathcal{F}\}.$$

Let $\mathcal{S}$ be the family of functions f meromorphic and univalent in D and such that

$$f(z) = a_{-1}z^{-1} + a_0 + a_1 z + \ldots, \qquad z \in D,$$

where $a_n = a_n(f)$ $(n \geqslant -1)$ are complex constants. We shall investigate $\omega(\mathcal{F})$ and $\omega(\mathcal{F}')$ for some subfamilies $\mathcal{F}$ of $\mathcal{S}$.

Let S be the family of all holomorphic members of $\mathcal{S}$:

$$S = \{f \in \mathcal{S}; a_{-1}(f) = 0\}.$$

Piranian [3] proved that $\omega(S') = 3$. Therefore, a natural question arises:

$$\omega(\mathcal{S}') < +\infty? \tag{1.1}$$

We shall show among other things that (1.1) is answered in the negative. For each $k \geqslant 0$ we now set

$$\mathcal{S}_k = \{f \in \mathcal{S}; |a_{-1}(f)| > k\},$$

to propose

THEOREM. *We have*

$$\omega(\mathcal{S}_k) = +\infty \quad \textit{for each } k \geqslant 0. \tag{1.2}$$

Furthermore,

$$\omega(\mathcal{S}'_0) = +\infty, \tag{1.3}$$

while for each $k > 0$

$$b(k) \leqslant \omega(\mathcal{S}'_k) \leqslant c(k) < +\infty, \tag{1.4}$$

where

$$b(k) = \frac{12k}{9k^2+1}, \quad c(k) = \frac{3k+10}{\sqrt{k(k+2)}}.$$

Since $\mathcal{S}' = \mathcal{S}'_0 \cup S'$, it follows from (1.3) that $\omega(\mathcal{S}') = +\infty$ or (1.1) is answered in the negative. However, $\omega(g) < +\infty$ for each $g \in \mathcal{S}'$. For the proof, we choose $f \in \mathcal{S}$ with $g = f'$. Then f is normal (and univalent) in D. According to [6] (p. 141-142, Remark), f' is normal in D, whence $\omega(g) < +\infty$.

We note that $\omega(S) = +\infty$ is easily proved. For the proof we consider $f_n(z) = nz$, $z \in D$, $n = 1, 2, \ldots$ Then $f_n \in S$ with

$$\omega(S) \geqslant \omega(f_n) \geqslant f_n^{\#}(0) = n \to +\infty.$$

2. Proof of the Theorem. To prove (1.2) we set $p = 1/\sin(a/2)$ for $0 < a < \pi$, and

$$\begin{aligned} f_p(z) &= (k+1)p(1+pz)/[z(z+p)] \\ &= (k+1)[z^{-1} + (p-p^{-1}) + \ldots], \quad z \in D. \end{aligned}$$

Then f_p maps D one-to-one onto the extended plane $C^* = \{|z| \leqslant +\infty\}$ slit along the circular arc

$$\{(k+1)pe^{it};\ -a \leqslant t \leqslant a\}$$

(see [4], p. 14-15, Example 1.3, with a few modifications). Since

$$f_p(-p^{-1}) = 0 \quad \text{and} \quad f'_p(-p^{-1}) = (k+1)p^4/(1-p^2),$$

it follows that

$$\omega(f_p) \geqslant (1-p^{-2}) f_p^{\#}(-p^{-1}) = (k+1)p^2 \to +\infty$$

as $a \to 0$. Since $f_p \in \mathcal{S}_k$, we have

$$\omega(\mathcal{S}_k) = +\infty \quad (k \geqslant 0).$$

For the proof of (1.3) we set

$$f_a(z) = a^2(z+z^{-1})/(a^2-1), \quad 0 < a < 1,\ z \in D.$$

Since $f_a \in \mathscr{S}_0$ with

$$f_a'(a) = 1 \quad \text{and} \quad f_a''(a) = 2/[a(a^2-1)],$$

it follows that

$$\omega(f_a') \geqslant (1-a^2)f_a'^{\#}(a) = a^{-1} \to +\infty$$

as $a \to 0$. We thus obtain (1.3).

For the proof of (1.4) we need a lemma, where $\lambda(g) = g''/g'$ is used for g meromorphic in a domain in C^*.

LEMMA. *For each $g \in \mathscr{S}$ with $a_{-1}(g) = 1$, we have the estimates*

$$(2.1) \qquad (1-|z|^2)|z\lambda(g)(z)| \leqslant 10-4|z|^2, \quad z \in D;$$

$$(2.2) \qquad 1-|z|^2 \leqslant |z^2 g'(z)|, \quad z \in D.$$

To prove (2.1) we apply the inequality

$$(2.3) \qquad \big|w\lambda(F)(w)+(4|w|^2-2)/(|w|^2-1)\big| \leqslant 4|w|^2/(|w|^2-1),$$
$$1 < |w| \leqslant +\infty,$$

to F, defined by $F(w) = g(w^{-1})$. Estimate (2.3) follows from (29) in Theorem 4 of [1], p. 139, by eliminating the terms containing E and K in its both sides. Since

$$\lambda(F)(w) = -w^{-2}\lambda(g)(w^{-1})-2w^{-1},$$

it follows from (2.3) with $z = w^{-1}$ that

$$\big|-z\lambda(g)(z)-2+(4-2|z|^2)/(1-|z|^2)\big| \leqslant 4/(1-|z|^2),$$

whence

$$(1-|z|^2)|z\lambda(g)(z)| \leqslant 8-2|z|^2+2(1-|z|^2) = 10-4|z|^2$$

for $z \in D$.

For the proof of (2.2) we apply to F a part of the Loewner distortion theorem (see [1], p. 133, (15)):

$$1-|w|^{-2} \leqslant |F'(w)|.$$

Then (2.2) follows by setting $w^{-1} = z \in D$.

We now prove (1.4). For each $f \in \mathscr{S}_k$ we set $g = f/a_{-1}(f)$, so that our Lemma may be applied to g with $\lambda(f) = \lambda(g)$. It then follows from (2.1) that

$$(2.4) \qquad (1-|z|^2)|\lambda(f)(z)| \leqslant (10-4|z|^2)/|z|, \quad z \in D-\{0\},$$

whence

$$(2.5) \qquad (1-|z|^2)f'^{\#}(z) \leqslant \tfrac{1}{2}(1-|z|^2)|\lambda(f)(z)| \leqslant (5-2|z|^2)/|z| \equiv \varphi(|z|)$$

for $z \in D-\{0\}$. On the other hand, it follows from (2.2) that

$$k(1-|z|^2)/|z|^2 < |a_{-1}(f)|(1-|z|^2)/|z|^2 \leqslant |f'(z)|,$$

whence

$$|f'(z)|/(1+|f'(z)|^2) < 1/|f'(z)| < |z|^2/[k(1-|z|^2)] \tag{2.6}$$

for $z \in D-\{0\}$. Multiplying both sides of (2.4) and (2.6) we obtain

$$(1-|z|^2)f'^{\#}(z) \leqslant |z|(10-4|z|^2)/[k(1-|z|^2)] \equiv \psi(|z|) \tag{2.7}$$

for $z \in D$; in effect, $f'^{\#}(0) = 0$ since f' has 0 as the double pole. Now the function φ of $|z| \in (0,1)$ in (2.5) decreases from $+\infty$ to 3 as $|z|$ increases from 0 to 1, while the function ψ of $|z| \in [0,1)$ in (2.7) increases from 0 to $+\infty$ as $|z|$ increases from 0 to 1. Therefore

$$\varphi(|z|) = \psi(|z|) = c(k) \quad \text{only for } |z| = \gamma(k) \equiv \sqrt{k/(k+2)}.$$

We thus obtain

$$\begin{aligned} &\varphi(|z|) \leqslant c(k) \quad \text{for } \gamma(k) \leqslant |z| < 1, \\ &\psi(|z|) \leqslant c(k) \quad \text{for } |z| \leqslant \gamma(k). \end{aligned} \tag{2.8}$$

Combining (2.5), (2.7) and (2.8) we now conclude that

$$\omega(f') \leqslant c(k), \quad f \in \mathscr{S}_k,$$

whence

$$\omega(\mathscr{S}'_k) \leqslant c(k).$$

We next consider the function

$$f_\varepsilon(z) = (k+\varepsilon)(z+z^{-1}) \quad (k>0,\ \varepsilon>0).$$

Then $f_\varepsilon \in \mathscr{S}_k$ with $f'_\varepsilon(\frac{1}{2}) = -3(k+\varepsilon)$ and $f''_\varepsilon(\frac{1}{2}) = 16(k+\varepsilon)$. Therefore

$$\omega(\mathscr{S}'_k) \geqslant (1-(\tfrac{1}{2})^2)f'^{\#}_\varepsilon(\tfrac{1}{2}) = 12(k+\varepsilon)/[1+9(k+\varepsilon)^2].$$

Letting $\varepsilon \to 0$ we now obtain

$$b(k) \leqslant \omega(\mathscr{S}'_k),$$

which completes the proof of (1.4).

Finally, we propose two problems:

P 1199. What is the precise value of $\omega(\mathscr{S}'_k)$ $(k>0)$?

P 1200. Since $\omega(\mathscr{S}'_k)$ decreases as $k>0$ increases, it is of interest, at least, to find

$$\lim_{k\to+\infty} \omega(\mathscr{S}'_k).$$

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