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A NOTE ON THE MARCINKIEWICZ INTEGRAL

BY

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In their work on Fourier series Littlewood and Paley, and in his work on the boundary values of analytic functions Lusin, introduced the well-known g and g_λ^*, and the area S functions, respectively. In this context, Marcinkiewicz [4] considered the expression $\mu(f)(x)$ given by

$$\mu(f)(x) = \Big(\int_{[0,2\pi]} \frac{|F(x+t) + F(x-t) - 2F(x)|^2}{t^3}\, dt\Big)^{1/2}, \quad x \in [0, 2\pi],$$

where $F(x) = \int_{[0,x]} f(t)\, dt$. The Marcinkiewicz integral $\mu(f)(x)$ was introduced in order to give an analogue of the Littlewood–Paley g function without going into the interior of the unit disk for its definition; that there are similar results along the lines of the S and g_λ^* functions is exemplified by Theorem 5 below. It was Zygmund [7] who, among other interesting results, proved that

$$\|\mu(f)\|_p \le c_p \|f\|_p, \quad 1 < p < \infty.$$

Stein [5] defined a generalization of the Marcinkiewicz integral to higher dimensions, and proved similar results by means of the so-called real variables method, in the following setting. Let $\Omega(x)$ be a function which is homogeneous of degree 0 and which, in addition, satisfies the following two conditions:

(i) $\Omega(x)$ is continuous on Σ, the unit sphere of $\mathbf{R}^n$, and satisfies a Lipschitz condition of order α there, i.e.,

$$|\Omega(x') - \Omega(y')| \le c|x' - y'|^\alpha, \quad x', y' \in \Sigma.$$

(ii) $\int_\Sigma \Omega(x')\, dx' = 0$.

For a locally integrable function f on $\mathbf{R}^n$ and $t > 0$, let $F_t(f, x) = F_t(x)$ be given by

$$F_t(x) = \int_{\{|y|\le t\}} \frac{\Omega(y)}{|y|^{n-1}} f(x-y)\, dy, \quad x \in \mathbf{R}^n,$$

and define now $\mu(f)(x)$ by

$$\mu(f)(x) = \Big(\int_{[0,\infty)} \frac{|F_t(x)|^2}{t^3}\, dt\Big)^{1/2} .$$

Stein showed that if $f \in L^p(\mathbf{R}^n)$, $1 \le p \le 2$, then

$$\|\mu(f)\|_p \le c_p \|f\|_p , \quad 1 < p \le 2 , \tag{1}$$

and when $p = 1$,

$$\lambda |\{\mu(f) > \lambda\}| \le c\|f\|_1 , \quad \text{all } \lambda > 0 .$$

Benedek, Calderón and Panzone [1] showed that if Ω is continuously differentiable in $x \ne 0$, then (1) above holds for $1 < p < \infty$.

In order to state our first result we set

$$M_p f(x) = \sup_{x\in Q} \Big(|Q|^{-1} \int_Q |f(y)|^p\, dy\Big)^{1/p} ,$$

where Q is a cube containing x with sides parallel to the coordinate axes; this is the generalized Hardy–Littlewood maximal function. We put $M_1 f = Mf$.

We also need the generalized sharp maximal function $M_p^{\#} f$ given by

$$M_p^{\#} f(x) = \sup_{x\in Q} \Big(|Q|^{-1} \int_Q |f(y) - f_Q|^p\, dy\Big)^{1/p} ,$$

where f_Q is the average of f over Q. We put $M_1^{\#} f = M^{\#}$, and we set $\mathrm{BMO}(\mathbf{R}^n) = \{f : \|f\|_* = \|M^{\#} f\|_\infty < \infty\}$; by the John–Nirenberg inequality the expressions $\|M_p^{\#} f\|_\infty$ all give equivalent BMO norms for a given function f.

The first result we prove is

THEOREM 1. *Suppose $1 < p < \infty$, and that $\|\mu(f)\|_p \le k_p \|f\|_p$. Then there is a constant $c_p = c_p(k_p)$ independent of f such that*

$$M^{\#}(\mu(f))(x) \le c_p M_p f(x) , \quad \textit{for all } x \in \mathbf{R}^n .$$

Our next result deals with a commutator of the Marcinkiewicz integral, and for this purpose we take the point of view of the vector-valued singular integral operators of Benedek, Calderón and Panzone. Let, then, H be the Hilbert space

$$H = \Big\{ h : \|h\| = \Big(\int_{[0,\infty)} \frac{|h(t)|^2}{t^3}\, dt\Big)^{1/2} < \infty \Big\} .$$

For each fixed $x \in \mathbf{R}^n$, we may view $F_t(x)$ as a mapping from $[0,\infty)$ to H, and it is clear that $\|F_t(f,x)\| = \mu(f)(x)$.

For $b \in \mathrm{BMO}(\mathbf{R}^n)$, $C_b(f)(x)$, the commutator of the Marcinkiewicz integral, is then defined as

$$C_b(f)(x) = \|b(x)F_t(f,x) - F_t(bf,x)\|, \quad x \in \mathbf{R}^n.$$

We then have

THEOREM 2. *Given $1 < r,\ s < \infty$, there is a constant $c = c_{r,s}$ independent of b and f such that*

$$M^{\#}(C_b(f))(x) \le c\|b\|_*(M_r(\mu(f))(x) + M_s f(x)).$$

Theorems 1 and 2 lead to various weighted $L^p(\mathbf{R}^n)$ inequalities; we list some but do not prove them as the proof technique, once we have the pointwise estimates at hand, is by now well known.

THEOREM 3. *Let $1 < p < \infty$, and w a weight in the Muckenhoupt $A_p(\mathbf{R}^n)$ class. Then there is a constant $c = c_p(w)$ independent of f such that*

$$\|\mu(f)\|_{L^p_w} \le c\|f\|_{L^p_w}, \quad 1 < p < \infty.$$

THEOREM 4. *Let $1 < p < \infty$, and w a weight in the Muckenhoupt $A_p(\mathbf{R}^n)$ class. Then there is a constant $c = c_p(w)$ independent of f and b such that*

$$\|C_b(f)\|_{L^p_w} \le c\|b\|_*\|f\|_{L^p_w}, \quad 1 < p < \infty.$$

Finally, we consider the Marcinkiewicz integrals $\mu_S(f)$ and $\mu^*_\lambda(f)$ corresponding to the S and g^*_λ functions; they are defined by

$$\mu_S(f)(x)^2 = \int_{\Gamma(x)} \frac{|F_t(y)|^2}{t^{n+3}}\,dy\,dt,$$

where $\Gamma(x) = \{(y,t) : |x-y| < t\}$, and

$$\mu^*_\lambda(f)(x)^2 = \int_{\mathbf{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} \frac{|F_t(y)|^2}{t^{n+3}}\,dy\,dt, \quad \lambda > 1,$$

respectively. As an indication that the theory in this case proceeds along the lines of the S and g^*_λ, and μ, functions, we show a result that includes the $L^2(\mathbf{R}^n)$ continuity case.

THEOREM 5. *Suppose w is a nonnegative locally integrable function in $\mathbf{R}^n$. Then there is a constant c independent of f and w such that*

$$\int_{\mathbf{R}^n} \mu^*_\lambda(f)(x)^2 w(x)\,dx \le c \int_{\mathbf{R}^n} |f(x)|^2 Mw(x)\,dx.$$

We pass now to the proofs.

Proof of Theorem 1. Given $x \in \mathbf{R}^n$, let $Q = Q(x_0, h)$ be a cube centered at x_0 of edgelength h with sides parallel to the axes. If now $Q^* =$

$Q(x_0, 4\sqrt{n}h)$, let

$$f = f\chi_{Q^*} + f(1-\chi_{Q^*}) = f_1 + f_2\,,$$

say. Then

$$\begin{aligned}\int_Q \mu(f_1)(y)^p\,dy &\le \int_{\mathbf{R}^n} \mu(f_1)(y)^p\,dy\\ &\le c_p \int_{\mathbf{R}^n} |f_1(y)|^p\,dy \le c_p \int_{Q^*} |f(y)|^p\,dy\,,\end{aligned}$$

and consequently,

$$|Q|^{-1}\int_Q \mu(f_1)(y)\,dy \le \Big(|Q|^{-1}\int_Q \mu(f_1)(y)^p\,dy\Big)^{1/p} \le c_p M_p f(x)\,. \tag{2}$$

Next, given $w \in Q$, we estimate $I = |\mu(f_2)(x_0) - \mu(f_2)(w)|$. From the inequality

$$\big|\,\|F_t(f_2,x_0)\| - \|F_t(f_2,w)\|\,\big| \le \|F_t(f_2,x_0) - F_t(f_2,w)\|\,,$$

it readily follows that I does not exceed $I_1 + I_2 + I_3$, where

$$I_1 = \Big(\int_{[0,\infty)} \frac{1}{t^3}\Big(\int_{\{|x_0-y|<t\le|w-y|\}} \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}|f_2(y)|\,dy\Big)^2 dt\Big)^{1/2}\,,$$

$$I_2 = \Big(\int_{[0,\infty)} \frac{1}{t^3}\Big(\int_{\{|w-y|<t<|x_0-y|\}} \frac{|\Omega(w-y)|}{|w-y|^{n-1}}|f_2(y)|\,dy\Big)^2 dt\Big)^{1/2}\,,$$

and I_3 equals

$$\Big(\int_{[0,\infty)} \frac{1}{t^3}\Big(\int_{\{|x_0-y|\le t,|w-y|\le t\}} \Big|\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} - \frac{\Omega(w-y)}{|w-y|^{n-1}}\Big||f_2(y)|\,dy\Big)^2 dt\Big)^{1/2}.$$

Since the estimates for I_1 and I_2 follow along similar lines, we only consider I_1 here. Since Ω is bounded and since $|x_0-y| \sim |x-y|$, by Minkowski's inequality, I_1 does not exceed

$$\begin{aligned} & c\int_{\mathbf{R}^n} \frac{|f_2(y)|}{|x_0-y|^{n-1}}\Big(\int_{(|x_0-y|,|w-y|]} \frac{1}{t^3}\,dt\Big)^{1/2} \\ &\le c\int_{\mathbf{R}^n\setminus Q^*} \frac{|f(y)|}{|x_0-y|^{n-1}}\Big(\frac{1}{|x_0-y|^2} - \frac{1}{|w-y|^2}\Big)^{1/2} dy \\ &\le ch^{1/2}\int_{\mathbf{R}^n\setminus Q^*} \frac{|f(y)|}{|x_0-y|^{n-1}}\frac{1}{|x_0-y|^{3/2}}\,dy \\ &\le ch^{1/2}\int_{\mathbf{R}^n\setminus Q^*} \frac{|f(y)|}{|x-y|^{n+1/2}}\,dy\,. \end{aligned} \tag{3}$$

It is well known (cf. Torchinsky [6, p. 83]) that the above expression does not exceed $cMf(x) \le cM_p f(x)$, $1 < p < \infty$. Thus

$$I_1, I_2 \le cM_p f(x). \tag{4}$$

In order to estimate I_3 recall that $x_0, w \in Q$, $y \notin Q^*$, and note that

$$\begin{aligned}\left|\frac{\Omega(x_0-y)}{|x_0-y|^{n-1}} - \frac{\Omega(w-y)}{|w-y|^{n-1}}\right| &\le |\Omega(x_0-y)|\left|\frac{1}{|x_0-y|^{n-1}} - \frac{1}{|w-y|^{n-1}}\right| \\ &\quad + \frac{1}{|w-y|^{n-1}}|\Omega(x_0-y)-\Omega(w-y)| \\ &= A+B,\end{aligned}$$

say. By the mean value theorem, and since $|x-y| \sim |x_0-y| \sim |w-y|$, it readily follows that

$$A \le c\frac{|x_0-w|\,|x_0-y|^{n-2}}{|x_0-y|^{n-1}|w-y|^{n-1}} \le c\frac{|x_0-w|}{|x_0-y|^n} \le c\frac{h}{|x-y|^n}.$$

Similarly, since

$$\begin{aligned}|\Omega(x_0-y)-\Omega(w-y)| &= \left|\Omega\Big(\frac{x_0-y}{|x_0-y|}\Big) - \Omega\Big(\frac{w-y}{|w-y|}\Big)\right| \\ &\le c\left|\frac{x_0-y}{|x_0-y|} - \frac{w-y}{|w-y|}\right|^\alpha \le c\frac{|x_0-w|^\alpha}{|x_0-y|^\alpha},\end{aligned}$$

we also have

$$B \le c\frac{h^\alpha}{|x-y|^{n-1+\alpha}}.$$

Thus, again by Minkowski's inequality, it follows that I_3 does not exceed (a multiple of)

$$\begin{aligned}h\int_{\mathbf{R}^n} \frac{|f_2(y)|}{|x-y|^n}\Big(\int_{\{|x_0-y|\le t, |w-y|\le t\}} \frac{1}{t^3}\,dt\Big)^{1/2} dy \\ + h^\alpha \int_{\mathbf{R}^n} \frac{|f_2(y)|}{|x-y|^{n-1+\alpha}}\Big(\int_{\{|x_0-y|\le t, |w-y|\le t\}} \frac{1}{t^3}\,dt\Big)^{1/2} dy.\end{aligned}$$

Since $|x_0-y| \sim |x-y| \sim |w-y|$, the innermost integrals involving t above are of order $|x-y|^{-1}$. Consequently,

$$I_3 \le ch\int_{\mathbf{R}^n\setminus Q^*} \frac{|f(y)|}{|x-y|^{n+1}}\,dy + ch^\alpha \int_{\mathbf{R}^n\setminus Q^*} \frac{|f(y)|}{|x-y|^{n+\alpha}}\,dy. \tag{5}$$

Again by an argument similar to the one used in bounding I_1 above, the right-hand side of (5) does not exceed $cMf(x) \le cM_p f(x)$. Hence, combining the above estimates, it follows that

$$|\mu(f_2)(x_0) - \mu(f_2)(w)| \le cM_p f(x), \qquad \text{for all } w \in Q,$$

and consequently,

$$|Q|^{-1}\int_Q |\mu(f_2)(w)-\mu(f_2)(x_0)|\,dw \le cM_pf(x)\,. \tag{6}$$

Finally, since

$$\begin{aligned}|\mu(f_1+f_2)(w)&-\mu(f_2)(x_0)|\\ &\le \mu(f_1)(w)+|\mu(f_2)(w)-\mu(f_2)(x_0)|\,,\qquad \text{for all } w\in Q\,,\end{aligned}$$

by (2) and (6) above it follows that

$$|Q|^{-1}\int_Q |\mu(f)(w)-\mu(f_2)(x_0)|\,dw \le cM_pf(x)\,,$$

and the proof is complete. ■

Proof of Theorem 2. The proof follows along the lines of that of Theorem 1. Given a point $x\in \mathbf{R}^n$, suppose $Q=Q(x_0,h)$ is a cube containing it, and put $Q^*=Q(x_0,4\sqrt{n}h)$. If b_Q denotes the average of b over the cube Q, note that

$$\begin{aligned}b(y)F_t(f,y)&-F_t(bf,y)\\ &=(b(y)-b_Q)F_t(f,y)+b_QF_t(f,y)-F_t(bf,y)\\ &=(b(y)-b_Q)F_t(f,y)-F_t((b-b_Q)f,y)=A+B\,,\end{aligned}$$

say. First we estimate the average of $\|A\|\le |b(y)-b_Q|\,\|F_t(f)\|$ over Q. By Hölder's inequality with indices $1<r,\ r'<\infty$, and the John–Nirenberg inequality, it does not exceed

$$\begin{aligned}\Big(|Q|^{-1}\int_Q |b(y)-b_Q|^{r'}\,dy\Big)^{1/r'}&\Big(|Q|^{-1}\int_Q \|F_t(f)(y)\|^{r}\,dy\Big)^{1/r}\\ &\le c\|b\|_*\inf_{y\in Q} M_r(\mu(f))(y)\,.\end{aligned} \tag{7}$$

To bound the average of $\|B\|$ over Q note that

$$\|B\|\le \|F_t((b-b_Q)\chi_{Q^*}f,y)\|+\|F_t((b-b_Q)\chi_{\mathbf{R}^n\setminus Q^*}f,y)\|=\|B_1\|+\|B_2\|\,,$$

say. Let $1<q,\ u<\infty$ be such that $qu=s$. Then by Hölder's inequality and the boundedness of the Marcinkiewicz integral in $L^q(\mathbf{R}^n)$ it follows that

$$\begin{aligned}|Q|^{-1}\int_Q \|B_1\|\,dy &\le \Big(|Q|^{-1}\int_Q \|B_1\|^q\,dy\Big)^{1/q}\\ &\le c\Big(|Q|^{-1}\int_{Q^*}|b(y)-b_Q|^q|f(y)|^q\,dy\Big)^{1/q}\\ &\le c\Big(|Q^*|^{-1}\int_{Q^*}|b(y)-b_Q|^{qu'}\,dy\Big)^{1/u'}\Big(|Q^*|^{-1}\int_{Q^*}|f(y)|^{qu}\,dy\Big)^{1/qu}\end{aligned} \tag{8}$$

$$\le c\|b\|_* \inf_{y\in Q} M_s f(y)\,.$$

Finally, to bound $\|B_2\|$, as in the proof of Theorem 1, it suffices to estimate

$$\|F_t((b-b_Q)\chi_{\mathbf{R}^n\setminus Q^*}f, y) - F_t((b-b_Q)\chi_{\mathbf{R}^n\setminus Q^*}f, x_0)\|\,.$$

By the argument in the proof of the theorem, with $f_2(y)$ assuming the value $(b(y)-b_Q)\chi_{\mathbf{R}^n\setminus Q^*}(y)f(y)$ now, we estimate the above expression by a sum of three terms, corresponding to I_1, I_2 and I_3, respectively. By estimate (3), then, the terms corresponding to I_1 and I_2 are dominated by

$$ch^{1/2}\int_{\mathbf{R}^n\setminus Q^*}\frac{|b(y)-b_Q|\,|f(y)|}{|x-y|^{n+1/2}}\,dy\,,$$

which, by Hölder's inequality with indices $1 < s$, $s' < \infty$, is in turn bounded by

$$c\Big(h^{s'/2}\int_{\mathbf{R}^n\setminus Q^*}\frac{|b(y)-b_Q|^{s'}}{|x-y|^{s'(n+1/2)}}\,dy\Big)^{1/s'}\Big(h^{s/2}\int_{\mathbf{R}^n\setminus Q^*}\frac{|f(y)|^{s}}{|x-y|^{s(n+1/2)}}\,dy\Big)^{1/s}.$$

The second term above does not exceed $cM_sf(x)$. Similarly, by a result of Fefferman and Stein [3], the first term does not exceed $cM^{\#}_{s'}b(x) \le c\|b\|_*$.

In order to estimate the term corresponding to I_3, we make use of the estimate (5). Thus, this term does not exceed

$$ch\int_{\mathbf{R}^n\setminus Q^*}\frac{|b(y)-b_Q|\,|f(y)|}{|x-y|^{n+1}}\,dy + ch^{\alpha}\int_{\mathbf{R}^n\setminus Q^*}\frac{|b(y)-b_Q|\,|f(y)|}{|x-y|^{n+\alpha}}\,dy\,.$$

Hence, as above, this term is also bounded by $c\|b\|_*M_sf(x)$. It is now a simple matter, left to the reader, to complete the proof. ∎

To close this paper, and show how these results extend to the more general Marcinkiewicz integrals introduced above, we prove Theorem 5.

Proof of Theorem 5 (cf. Chanillo–Wheeden [2]). Clearly

$$\int_{\mathbf{R}^n}\mu^*_\lambda(f)(x)^2w(x)\,dx$$

$$= \int_{\mathbf{R}^{n+1}_+}|F_t(y)|^2\Big(\frac{1}{t^n}\int_{\mathbf{R}^n}w(x)\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}dx\Big)\,\frac{dy\,dt}{t^3}\,.$$

If A_k denotes the set

$$\Big\{(y,t)\in\mathbf{R}^{n+1}_+ : 2^{k-1} < \frac{1}{t^n}\int_{\mathbf{R}^n}w(x)\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda}dx \le 2^k\Big\}\,,$$

$$k = 0, \pm1, \dots,$$

the above expression is bounded by

$$\sum_k 2^k \int_{\mathbf{R}^{n+1}_+} |F_t(y)|^2 \chi_{A_k}(y,t) \frac{dy\,dt}{t^3}.$$

Now, note that if $(y,t) \in A_k$, then also

$$\frac{1}{t^n} \int_{\mathbf{R}^n} w(x)\Big(\frac{t}{t+|x-y|}\Big)^{n\lambda} dx \le cMw(y).$$

Now, if $|y-z| < t$, then $t+|x-y| \sim t+|x-z|$, and consequently, if $|y-z| < t$ and $(y,t) \in A_k$, then

$$2^{k-1} \le c\frac{1}{t^n} \int_{\mathbf{R}^n} w(x)\Big(\frac{t}{t+|x-z|}\Big)^{n\lambda} dx \le cMw(z).$$

In particular, if $(y,t) \in A_k$ and $|y-z| < t$, then $z \in E_k = \{z \in \mathbf{R}^n : Mw(z) \ge c2^k\}$. Thus, for $(y,t) \in A_k$ and $z \in \mathbf{R}^n$ such that $|y-z| \le t$, we have $f(z) = f(z)\chi_{E_k}(z)$, and consequently, $F_t(f,y) = F_t(f\chi_{E_k},y)$. Hence it follows that

$$\begin{aligned}\int_{\mathbf{R}^{n+1}_+} |F_t(y)|^2 \chi_{A_k}(y,t) \frac{dy\,dt}{t^3} &= \int_{\mathbf{R}^{n+1}_+} |F_t(f\chi_{E_k},y)|^2 \chi_{A_k}(y,t) \frac{dy\,dt}{t^3} \\ &\le \int_{\mathbf{R}^n} \mu(f\chi_{E_k},y)^2\, dy \le c\|f\chi_{E_k}\|_2^2,\end{aligned}$$

for all k. Thus,

$$\begin{aligned}\int_{\mathbf{R}^n} \mu_\lambda^*(f)(x)^2 w(x)\, dx &\le c\sum_k 2^k \|f\chi_{E_k}\|_2^2 \\ &= c\int_{\mathbf{R}^n} |f(x)|^2 \sum_k 2^k \chi_{E_k}(x)\, dx.\end{aligned}$$

By the definition of the sets E_k, the above sum does not exceed $cMw(x)$. Thus, the right-hand side above is bounded by $c\int_{\mathbf{R}^n} |f(x)|^2 Mw(x)\, dx$, and the proof is complete. ∎

Since $\mu_S(f)(x) \le c\mu_\lambda^*(f)(x)$, the same result holds for the Marcinkiewicz integral related to the Lusin function.

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