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ON A CARDINAL INVARIANT RESEMBLING THE SOUSLIN NUMBER

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0. In this paper the word *space* means a Tichonoff space. If X is a space, $Y \subseteq X$, then $\operatorname{Int} Y$ is the interior of Y. Here $Y \sim \emptyset$ or $Y_1 \sim Y_2$ means that $\operatorname{Int} Y = \emptyset$ or $\operatorname{Int}(Y_1 \setminus Y_2) = \emptyset$, respectively. The space $I = [0, 1]$ is the line segment with its natural topology, and I^τ denotes the Tichonoff cube of weight τ. By T_τ we denote the discrete space of cardinality τ, T_τ^* is its one-point bicompactification, and finally E^τ is the Kowalsky space of weight τ.

The letters U, V, W, O with or without indices denote only open subsets of X, the letters F, Φ, P stand only for closed ones. Let $\gamma = \{Y_\alpha : \alpha \in A\}$ be a system of subsets of X. For $Z \subseteq X$ let

$$\gamma \cap Z = \{\tilde{Y}_\alpha : \alpha \in A,\ \tilde{Y}_\alpha = Y_\alpha \cap Z\}.$$

The union of all elements of γ is denoted by $\bigcup \gamma$, and their intersection by $\bigcap \gamma$. The equality $|A| = \tau$ means that the cardinality of A equals τ. The least upper bound for the cardinalities of closed discrete subspaces of X is referred to as $\operatorname{ext}(X)$, and

$$\Psi(X) = \sup_{F \subseteq X} \psi(F, X).$$

0.1. Definition. Let $F \subseteq X$ and let $\gamma = \{U_\alpha : \alpha \in A\}$ be a family of neighbourhoods of F. The family γ is called *admissible for* F if $\bigcap \gamma \sim F$.

0.2. Definition. If X is a space and $F \subseteq X$, let $\lambda(F, X) \leqslant \tau$ if and only if there exists a family $\gamma = \{U_\alpha : \alpha \in A\}$ such that $F \subseteq U_\alpha$ for every $\alpha \in A$ and

(a) γ is admissible for F,

(b) $|\gamma| = |A| \leqslant \tau$.

0.3. Remark. From now on the invariant λ will be called a *quasicharacter of* F in X. The *quasicharacter of* X (denoted further by $\lambda(X)$) is the smallest cardinal number τ for which the inequality $\lambda(F, X) \leqslant \tau$ holds for every $F \subseteq X$. It is clear that $\lambda(X) \leqslant \Psi(X)$.

1. Formulating Definitions 0.1 and 0.2 in terms of complements we obtain the following

1.1. LEMMA. *For an arbitrary space* X *we have* $\lambda(X) \leqslant \tau$ *if and only if for every* $U \subseteq X$ *there exists a family* $\gamma = \{F_\alpha\colon \alpha \in A\}$ *such that*

(a) $F_\alpha \subseteq U$ *for every* $\alpha \in A$,

(b) $[\bigcup \gamma] \supseteq U$,

(c) $|\gamma| \leqslant \tau$.

A family γ satisfying (a) and (b) will be called *admissible for* U. If, moreover, (c) holds we write $\lambda(U, X) \leqslant \tau$.

It follows immediately from the lemma that $\lambda(U) \leqslant \lambda(X)$ if $U \subseteq X$, i.e., the quasicharacter is hereditary with respect to open subsets.

1.2. PROPOSITION. *Let* $f\colon X \to Y$ *be a closed continuous mapping of* X *onto* Y. *Then* $\lambda(Y) \leqslant \lambda(X)$.

Proof. If $V \subseteq Y$, then for $U = f^{-1} V$ there exists an admissible system $\gamma = \{F_\alpha\colon \alpha \in A\}$ of cardinality not greater than $\lambda(X)$. The family $f(\gamma) = \{fF_\alpha\colon \alpha \in A\}$ is evidently admissible for V. Clearly, $|f(\gamma)| \leqslant \lambda(X)$, and thereby $\lambda(Y) \leqslant \lambda(X)$.

1.3. THEOREM. *If* X *is a space, then* $\lambda(X) \leqslant c(X)$.

Proof. Take any $U \subseteq X$. Let $\gamma_1 = \{V_\alpha\colon \alpha \in A\}$ be a maximal disjoint family of F_σ-sets, contained in U. It is readily seen that $\bigcup \gamma_1$ is dense in U. Let

$$V_\alpha = \bigcup_{n \in N^+} F_\alpha^n,$$

where N^+ is the set of all positive integers. It is clear that $\gamma = \{F_\alpha^n\colon \alpha \in A,\ N^+ \ni n\}$ is admissible for U and

$$|\gamma| \leqslant |A| \cdot \aleph_0 \leqslant c(X).$$

The quasicharacter of X does not necessarily coincide with the Souslin number or with $\Psi(X)$. It is evident that $\lambda(T_\tau) \leqslant \aleph_0$ for every τ, while $c(T_\tau) = \tau$. The same holds for the space E^τ. We are going to give an example of a space X whose quasicharacter is strictly smaller than the Souslin number and the pseudocharacter at the same time. It can also be mapped openly onto a space Y such that $\lambda(Y) > \lambda(X)$.

1.4. EXAMPLE. Let $X = (T_\tau^* \times I) \backslash (p^* \times (0, 1])$, where p^* is a single unisolated point of T_τ^*. It is clear that

$$X = (T_\tau \times I) \cup \{x^*\}, \quad \text{where } x^* = (p^*, 0).$$

It follows from the definition of X that $c(X) = \tau$ and $\psi(x^*, X) = \tau$. Let us verify that $\lambda(X) = \aleph_0$. If we are given $U \subseteq X$, let

$$V_n = (T_\tau^* \times [0, 1/n)) \cap X \text{ for all } n \in N^+ \quad \text{and} \quad W_n = V_n \backslash [V_{n+1}].$$

It is readily seen that $\bigcup_{n\in N^+} W_n$ is dense in X, so $\bigcup_{n\in N^+} (W_n \cap U)$ is dense in U. It is enough, therefore, to prove that $W_n \cap U$ is an F_σ-set for every $n \in N^+$. In fact, $W_n \cap U$ is an F_σ-set in $T_\tau \times I$ and $[W_n \cap U] \not\ni x^*$. As a closed subset of $T_\tau \times I$ is closed in X if and only if its closure in X does not contain x^*, it is proved that $W_n \cap U$ is an F_σ-set in X. If

$$W_n \cap U = \bigcup_{n\in N^+} F_n^i,$$

then $\gamma = \{F_n^i\colon i, n \in N^+\}$ is an admissible countable family for U. Now let $Y = T_\tau^*$ and let $\pi\colon X \to Y$ be the natural projection. It is not difficult to prove that π is open and, as all points of $T_\tau \subseteq T_\tau^* = Y$ are open subsets of Y, its quasicharacter coincides with its pseudocharacter equal to τ.

1.5. THEOREM. *For any space X, $c(X) \leqslant \lambda(X)\operatorname{ext}(X)$.*

Proof. Write $\lambda(X) = \lambda$, $\operatorname{ext}(X) = \tau$. Assume that $c(X) > \lambda\tau$. It is possible therefore to choose a disjoint family $\sigma = \{U_\alpha\colon \alpha \in A\}$ of cardinality τ_0 greater than $\lambda\tau$. Let γ be an admissible family for $F = X \setminus \bigcup \sigma$ such that $|\gamma| \leqslant \lambda$. As $\bigcap \gamma \sim F$, there exists an $x_\alpha \in U_\alpha \setminus \bigcap \gamma$ for every $\alpha \in A$. The immediate consequence of disjointness of σ is that $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$, and thereby the family $\sigma_1 = \{\{x_\alpha\}\colon \alpha \in A\}$ is of cardinality τ_0. Let $Y_0 = \bigcup \sigma_1$ and $Y = [Y_0]$. Evidently, Y_0 is an open discrete subset of Y, which in turn is closed in X. We shall show that $Y \setminus Y_0$ is a G_λ-set in Y. In fact, $Y \setminus Y_0 \subseteq F$, so $\gamma_1 = \gamma \cap Y$ is a family of neighbourhoods of $Y \setminus Y_0$ in Y. Remembering the way we have chosen the points x_α we obtain $\bigcap \gamma \cap Y_0 = \emptyset$. But

$$\bigcap \gamma_1 \cap Y_0 = \bigcap \gamma \cap Y \cap Y_0 = \emptyset.$$

Hence $\bigcap \gamma_1 = Y \setminus Y_0$ and our assertion is proved. To complete the proof of the theorem, observe that Y_0 is an F_λ-set in Y and, consequently, in X. But Y_0 is then a union of at most λ subspaces, closed in X and discrete by the discreteness of Y_0. The power of every element of this union does not exceed $\operatorname{ext}(X) = \tau$, and as there are at most λ elements, we have $\tau_0 = |Y_0| \leqslant \lambda\tau$, contradicting the fact that $\tau_0 > \lambda\tau$ by our assumption.

1.6. COROLLARY. *If* $\operatorname{ext}(X) = X_0$, *then* $\lambda(X) = c(X)$.

From the last equality it follows easily that for compact, Lindelöf and countably compact spaces $\lambda(X) = c(X)$, and also $c(X) \leqslant \Psi(X)$.

1.7. EXAMPLE. Let $X = I^\tau$. From Corollary 1.6 it follows easily that $\lambda(X) = c(X) = \aleph_0$. As for the space T_τ^*, again $c(T_\tau^*) = \lambda(T_\tau^*) = \tau$. Observing that $w(T_\tau^*) = \tau$ we see that there exists an embedding of T_τ^* into I^τ. This proves that the quasicharacter is not hereditary with respect to closed sets even in compact spaces.

1.8. PROPOSITION. *For a normal space X and its dense subspace Y, $\lambda(Y) \leqslant \lambda(X)$.*

Proof. We shall show that if $\lambda(X)=\tau$, then for every $U\subseteq X$ there exists an admissible set system γ such that $|\gamma|\leqslant\tau$ and $\bigcup\gamma$ is open in X. In fact, for any $F\in\gamma_0$, an arbitrary admissible family for U of cardinality not greater than τ, in view of the normality of X it is possible to choose an F_σ-neighbourhood $V\supseteq F$ and $V\subseteq U$. As

$$V=\bigcup_{i\in N^+}\Phi_F^i,$$

let $\gamma(F)=\{\Phi_F^i\colon i\in N^+\}$. Finally, the family $\gamma=\{\gamma(F)\colon F\in\gamma_0\}$ is evidently admissible for U and $|\gamma|\leqslant|\gamma_0|\cdot\aleph_0\leqslant\tau$.

Now, if U_1 is open in Y, let us find a $U\subseteq X$, $U\cap Y=U_1$ and an admissible family γ for U such that $\bigcup\gamma$ is open and $|\gamma|\leqslant\tau$. Consider the family $\gamma_1=\gamma\cap Y$. Clearly, $|\gamma_1|\leqslant\tau$. Let us verify that γ_1 is admissible for U_1. In fact,

$$[\bigcup\gamma_1]=[\bigcup\gamma\cap Y].$$

As $\bigcup\gamma$ is open,

$$[\bigcup\gamma\cap Y]=[\bigcup\gamma]\supseteq U\supseteq U_1.$$

The results below are concerned with some multiplicative properties of the quasicharacter. As there exists a large class of spaces X (with $\operatorname{ext}(X)=\aleph_0$) for which $\lambda(X)=c(X)$, the finite multiplicativity of the quasicharacter is independent of the set theory axioms. For the Souslin number this was proved in [1] and [2]. Juhász in [2] proved that the Souslin property holds for an arbitrary product of spaces if and only if every finite product of these spaces has the Souslin property. Kurepa proved in [3] that if $c(X_\alpha)\leqslant\tau$, $\alpha\in A$, then

$$c\Big(\prod_{\alpha\in A}X_\alpha\Big)\leqslant 2^\tau.$$

The author does not know whether the same results are true for the quasicharacter.

2. Now we shall prove several theorems, analogous to the above ones in some respect and among them an analogue of Marczewski's theorem [4].

2.1. THEOREM. *For any spaces X and Y,*

$$\lambda(Y\times X)\leqslant\min\{\lambda(X)d(Y),\ \lambda(Y)d(X)\}.$$

Proof. Let us prove, e.g., that $\lambda(Y\times X)\leqslant\lambda(X)d(Y)$. There exists a subset D of Y, dense in itself and of cardinality $d(Y)$. For a given set $U\subseteq Y\times X$ and for every $y\in D$ let

$$y(U)=(\{y\}\times X)\cap U.$$

As $\{y\}\times X$ is homeomorphic to X, the set $y(U)$ is homeomorphic to an open subset of X. Therefore there exists an admissible family γ_y for $y(U)$ in $\{y\}\times X$. The elements of γ_y are, evidently, closed in $Y\times X$ and $|\gamma_y|\leqslant\lambda(X)$.

Now, let

$$\gamma=\bigcup_{y\in D}\gamma_y.$$

It is clear that γ is admissible for U and $|\gamma|\leqslant\lambda(X)d(Y)$.

2.2. COROLLARY. *If X and Y are compact spaces, then*

$$c(Y\times X)\leqslant\min\{c(Y)d(X),\,c(X)d(Y)\}.$$

It is easy to verify that for any spaces X and Y the inequality $c(X\times Y)\geqslant c(X)$ holds. It is not known yet whether $\lambda(X\times Y)\geqslant\lambda(X)$ for every space Y, but under some restrictions on Y this can be proved.

2.3. PROPOSITION. *If Y has a point of local compactness, then $\lambda(X\times Y)\geqslant\lambda(X)$.*

Proof. Take a point $y_0\in Y$ and its neighbourhood O_{y_0} such that $[O_{y_0}]$ is compact. For any $F\subseteq X$ consider the product $F\times[O_{y_0}]$. Let γ_1 be its admissible family in $X\times Y$. Using

$$\gamma_2=\gamma_1\cap(X\times[O_{y_0}])$$

we shall construct an admissible family for F in X. As $[O_{y_0}]$ is compact, it is true for every $U\in\gamma_2$ that from $F\times[O_{y_0}]\subseteq U$ it follows that there exists a $V\subseteq X$ such that

$$F\times[O_{y_0}]\subseteq V\times[O_{y_0}]\subseteq U.$$

Let us denote such a V, depending on U, by $\psi(U)$. We assert finally that the family $\gamma=\{\psi(U)\colon U\in\gamma_2\}$ is admissible for F. In fact, if $W\subseteq\bigcap\gamma\setminus F$, then by the choice of γ we have

$$W_1=W\times[O_{y_0}]\subseteq U\quad\text{ for every } U\in\gamma_2.$$

Observing that $X\times O_{y_0}$ is dense in $X\times[O_{y_0}]$ we see that $W_1\cap(X\times O_{y_0})$ is not empty and open in $X\times Y$. It is also a subset of $\bigcap\gamma_1\setminus F$, contradicting the fact that γ_1 is admissible.

2.4. THEOREM. *For an arbitrary space X and for a metric space M we have $\lambda(X\times M)\leqslant\lambda(X)$.*

Proof. Taking a point from every element of a σ-discrete base of M we will obtain a sequence $\{\Phi_n\colon n\in N^+\}$ of closed discrete subspaces of M whose union is dense in M.

Let $U\subseteq X\times M$ and $U_y=(X\times\{y\})\cap U$ for all $y\in\bigcup_{n\in N^+}\Phi_n$. Let A be a set of indices, $|A|=\tau=\lambda(X)$. As U_y is open in $X\times\{y\}$, there exists an admissible family $\gamma^y = \{F_\alpha^y: \alpha \in A\}$ for U_y in $X \times \{y\}$. We shall show that the set

$$P_\alpha^n = \bigcup_{y \in \Phi_n} F_\alpha^y$$

is closed for every $\alpha \in A$ and $n \in N^+$. Let $z \in X \times M \setminus P_\alpha^n$. Then $z = (x, m)$, where $x \in X$ and $m \in M$. If $m \notin \Phi_n$, then

$$z \in X \times (M \setminus \Phi_n) \subseteq X \times M \setminus P_\alpha^n$$

because $P_\alpha^n \subseteq X \times \Phi_n$. If $m \in \Phi_n$, then $x \notin F_\alpha^m$ and, consequently, there exists a $V \subseteq X$ such that

$$(x, m) \in V \times \{m\} \subseteq X \times \{m\} \setminus F_\alpha^m.$$

It follows from the discreteness of Φ_n that we can choose a $W \subseteq M$ such that $W \cap \Phi_n = \{m\}$. It is clear that

$$(x, m) \in V \times W \subseteq X \times M \setminus P_\alpha^n$$

and it is proved that P_α^n is closed in $X \times M$. The family

$$\gamma = \{P_\alpha^n: \alpha \in A, n \in N^+\}$$

is, evidently, of power not greater than $|A| \cdot \aleph_0 = \tau$ and is admissible for U.

We are going to point out some classes of spaces for which the quasicharacter countability is multiplicative.

We will often use the following evident lemma:

2.5. LEMMA. *If γ is a σ-discrete family of open F_σ-sets, then $\bigcup \gamma$ is an F_σ-set.*

In fact, a countable sum of F_σ-sets is an F_σ-set and a union of a discrete family of F_σ-sets is an F_σ-set.

Let us introduce some auxiliary notation. The word *network* will mean a network of closed elements. Let M_α be a space for every $\alpha \in A$. Suppose that for all $\alpha \in A$ the space M_α has a base (a network, a π-base) which is σ-discrete. Denote this base (network, π-base) by B_α. A set $U \subseteq \prod_{\alpha \in A} M_\alpha$ is called *standard* if

$$U = \prod_{\alpha \in A} \pi_\alpha U,$$

where π_α is the natural projection of $M = \prod_{\alpha \in A} M_\alpha$ onto M_α and, besides, $\pi_\alpha U = M_\alpha$ for all $\alpha \in A$ except for a finite number of them, and if $\pi_\alpha U \neq M_\alpha$, then $\pi_\alpha U \in B_\alpha$. It is easy to prove that standard sets form a base (a network, a π-base) in M. If U is standard, then

$$\gamma(U) = \{\alpha \in A: \pi_\alpha U \neq M_\alpha\}.$$

If U and V are standard, $U \cap V = \emptyset$, then there exists an $\alpha \in \gamma(U) \cap \gamma(V)$ such that

$$\pi_\alpha U \cap \pi_\alpha V = \emptyset .$$

2.6. LEMMA. *The union of a disjoint family of standard elements is an F_σ-set.*

Proof. Take a family γ satisfying the hypothesis. As all elements of γ are standard,

$$\bigcup \gamma = \bigcup_{n \in N^+} (\bigcup \gamma_n),$$

where $\gamma_n = \{U \in \gamma \colon |\gamma(U)| = n\}$ for every $n \in N^+$. It follows from Lemma 2.5 that it is enough to prove that $\bigcup \gamma_n$ is an F_σ-set for all $n \in N^+$. We shall do this by induction. If $n = 1$, then for every $U \in \gamma_n$ we have $\gamma(U) = \{\alpha\}$, where $\alpha \in A$ is the same fixed index for all $U \in \gamma_n$. Consequently, the set

$$\bigcup \gamma_1 = \pi_\alpha^{-1}\Big(\bigcup_{U \in \gamma_1} \pi_\alpha U\Big)$$

is an F_σ-set as an inverse image of an F_σ-set. Assume that it is proved for all $n \leqslant k$ that $\bigcup \gamma_n$ is an F_σ-set. Verify that $\bigcup \gamma_{k+1}$ is of type F_σ. Take an arbitrary $U^* \in \gamma_{k+1}$ and let

$$\gamma(U^*) = \{\alpha_1, \ldots, \alpha_{k+1}\}.$$

For every $U \in \gamma_{k+1}$ we know that $\gamma(U) \cap \gamma(U^*) \neq \emptyset$; therefore

$$\bigcup \gamma_{k+1} = \bigcup_{i=1}^{k+1} (\bigcup \gamma_{k+1}^i), \quad \text{where } \gamma_{k+1}^i = \{U \in \gamma_{k+1} \colon \alpha_i \in \gamma(U)\}.$$

Applying again Lemma 2.5 observe that it is enough to show that $\bigcup \gamma_{k+1}^i$ is an F_σ-set for all $i \in \{1, \ldots, k+1\}$. But

$$\bigcup \gamma_{k+1}^i = \bigcup_{U \in B_{\alpha_i}} (\bigcup \gamma_{k+1,U}^i), \quad \text{where } \gamma_{k+1,U}^i = \{W \in \gamma_{k+1}^i \colon \pi_{\alpha_i} W = U\}.$$

Remembering that B_{α_i} is a σ-discrete base (network, π-base) and using for the last time Lemma 2.5, we see that it is enough to prove that, for all $U \in B_{\alpha_i}$, $\bigcup \gamma_{k+1,U}^i$ is an F_σ-set in M. But if $W \in \gamma_{k+1,U}^i$, then in the space $\prod_{\alpha \in A \setminus \{\alpha_i\}} M_\alpha \times U$, clearly, $|\gamma(W)| = k$. We may thus deduce from the induction assumption that $\bigcup \gamma_{k+1,U}^i$ is an F_σ-set in $\prod_{\alpha \in A \setminus \{\alpha_i\}} M_\alpha \times U$, and thereby an F_σ-set in M, because the subspace $\prod_{\alpha \in A \setminus \{\alpha_i\}} M_\alpha \times U$ is itself an F_σ-set in M.

2.7. THEOREM. *If for every $\alpha \in A$, M_α has a base (a π-base, a network) which is σ-discrete, then*

$$\lambda\Big(\prod_{\alpha \in A} M_\alpha\Big) = \aleph_0 .$$

Proof. Let

$$U \subseteq M = \prod_{\alpha \in A} M_\alpha.$$

Observe that standard sets form a base (a π-base, a network) in M. Therefore, if $\gamma = \{U_\beta\colon \beta \in B, U_\beta \subseteq U\}$ is a maximal disjoint family of standard elements, then $\bigcup \gamma$ is dense in U. By virtue of Lemma 2.6 we have

$$\bigcup \gamma = \bigcup_{n \in N^+} F_n,$$

so the family $\gamma_1 = \{F_n\colon n \in N^+\}$ is countable and admissible for U.

2.8. Corollary. *Let X be a φ-space, i.e., a space which is a closed image of a product of metric spaces. Then $\lambda(X) = \aleph_0$.*

This follows from Proposition 1.2 and Theorem 2.7.

The following problem was posed by Arhangel'skiĭ.

2.9. Problem. Let, for every $\alpha \in A$, M_α belong to exactly one of the following classes:

(a) closed images of metric spaces,

(b) stratifiable spaces,

(c) spaces having a σ-conservative network,

(d) φ-spaces.

Is it true that

$$\lambda\Big(\prod_{\alpha \in A} M_\alpha\Big) = \aleph_0?$$

Let X_α be a space for every $\alpha \in A$ and let

$$\lambda = \sup_{\alpha \in A} \lambda(X_\alpha), \qquad \tau = \sup_{\alpha \in A} \operatorname{ext}(X_\alpha), \qquad \mu = \sup_{\alpha \in A} c(X_\alpha).$$

2.10. Proposition. $\lambda\big(\prod_{\alpha \in A} X_\alpha\big) \leqslant 2^{\lambda\tau}$.

In fact,

$$\lambda\Big(\prod_{\alpha \in A} X_\alpha\Big) \leqslant c\Big(\prod_{\alpha \in A} X_\alpha\Big) \leqslant 2^\mu \leqslant 2^{\lambda\tau}$$

by Theorem 1.5 and Kurepa's result [3].

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