COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1977 FASC. 1

ON DOMAINS OF ATTRACTION OF RECORD VALUE DISTRIBUTIONS

BY

W. FREUDENBERG AND D. SZYNAL (LUBLIN)

1. Introduction and preliminaries. Let $\{X_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables with the common continuous distribution function F. X_j is said to be a *record value* of the sequence $\{X_n, n \geqslant 1\}$ if

$$X_j > \max\{X_1, X_2, \ldots, X_{j-1}\}.$$

By convention, X_1 is a record value. The random variables L_n, $n \geqslant 1$, defined by

$$L_1 = 1, \quad L_n = \min\{j : j > L_{n-1}, X_j > X_{L_{n-1}}\},\ n > 1,$$

give the indices at which record values occur.

The function R_F defined by

$$R_F(x) = -\log(1-F(x)), \quad -\infty \leqslant x \leqslant \infty,$$

is called the *hazard function* corresponding to the distribution function F.

The function

$$H_F(x) = 1-\exp\{-\sqrt{R_F(x)}\}, \quad -\infty \leqslant x \leqslant \infty,$$

is called the *associated distribution function* corresponding to F. The point

$$x_0^F = \sup\{x: F(x) < 1\}, \quad x_0 \leqslant \infty,$$

is said to be the *right end of the distribution function F*.

In what follows we shall write R, H, and x_0 instead of R_F, H_F, and x_0^F, respectively.

The limit laws for record values are very closely related to the limit laws for $M_n = \max\{X_1, X_2, \ldots, X_n\}$ of independent, identically distributed random variables $X_1, X_2, \ldots, X_n$ with the distribution function F, i.e. to the non-degenerate limit distributions G such that

$$\lim_{n\to\infty} \mathrm{P}[M_n \leqslant a_n x + b_n] = G(x), \tag{1}$$

where $a_n > 0$ and b_n, $n \geqslant 1$, are normalizing constants. The limit distributions satisfying (1), called the *extreme value distributions*, were fully characterized by Gnedenko [1]. The extreme value distributions belong to the type of one of the following three distributions:

$$\Lambda(x) = \exp\{-e^{-x}\}, \quad -\infty < x < \infty, \tag{2}$$

$$\Phi_a(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-a}\} & \text{if } x \geqslant 0, \end{cases} \tag{3}$$

$$\Psi_a(x) = \begin{cases} \exp\{-(-x)^a\} & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0, \end{cases} \tag{4}$$

where a is a positive constant.

If (1) holds, we write $F \in DM(G)$ (F belongs to the domain of attraction of the extreme value distribution G).

Similarly, the sequence $\{X_{L_n}, n \geqslant 1\}$ is said to have a *limit distribution* U if U is non-degenerate and there exist normalizing constants $a_n > 0$ and b_n, $n \geqslant 1$, such that

$$\lim_{n\to\infty} \mathrm{P}[X_{L_n} \leqslant a_n x + b_n] = U(x). \tag{5}$$

We call the distribution function U the *record value distribution* and we indicate (5) by writing $F \in DR(U)$ or, equivalently, $R \in DR(U)$, where R is the hazard function corresponding to F. Resnick [3] has proved that the record value distributions are of the form $\Phi\big(-2\log(-\log G)\big)$, where Φ denotes the standard normal distribution, and G is an extreme value distribution. Therefore, if $F \in DR(U)$, then U must be of the type of one of the following three distributions:

$$\Phi(x), \tag{6}$$

$$\Phi_{1,a}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \Phi(\log x^a) & \text{if } x \geqslant 0, \end{cases} \tag{7}$$

$$\Phi_{2,a}(x) = \begin{cases} \Phi\big(\log(-x)^{-a}\big) & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0, \end{cases} \tag{8}$$

where a is a positive constant.

In what follows we often need the theorems proved by Resnick in [2] and [3].

THEOREM 1 (Duality Theorem). *We have*

$$F \in DR(U) \quad \textit{iff} \quad H \in DM(G),$$

where $U = \Phi\big(-2\log(-\log G)\big)$.

THEOREM 2. *Let F and G be distribution functions with $x_0^F = x_0^G = x_0$, $x_0 \leqslant \infty$, and*

$$\lim_{x \to x_0-} \frac{1-F(x)}{1-G(x)} = a, \quad 0 < a < \infty. \tag{9}$$

If there exist normalizing constants $a_n > 0$ and b_n, $n \geqslant 1$, such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = V(x),$$

where V is non-degenerate, then

$$\lim_{n\to\infty} G^n(a_n x + b_n) = V^{1/a}(x).$$

The distribution functions F and G with the same right end satisfying (9) are called *a-tail equivalent.*

THEOREM 3. *Let F and G be distribution functions such that there exist normalizing constants $a_n > 0$ and b_n, $n \geqslant 1$, for which*

$$\lim_{n\to\infty} F^n(a_n x + b_n) = U(x), \quad \lim_{n\to\infty} G^n(a_n x + b_n) = V(x),$$

where U and V are extreme value distributions.

Then there exists β, $0 < \beta < \infty$, such that F and G are β-tail equivalent and there exist $A > 0$ and B such that $V(x) = U(Ax+B)$, where

(a) *if $U(x) = \Lambda(x)$, then $x_0 \leqslant \infty$, $A = 1$, and $\beta = e^B$;*
(b) *if $U(x) = \Phi_a(x)$, then $x_0 = \infty$, $B = 0$, and $\beta = A^a$;*
(c) *if $U(x) = \Psi_a(x)$, then $x_0 < \infty$, $B = 0$, and $\beta = A^{-a}$.*

If there exist constants $A > 0$ and B such that $V(x) = U(Ax+B)$, then we say that *U and V belong to the same class.*

Theorems 2 and 3 determine the domains of attraction of extreme value distributions by means of a-tail equivalence.

The aim of this note is to characterize the domains of attraction of record value distributions.

2. Hazard equivalence and its applications. In our considerations we often need the following obvious lemma.

LEMMA 1. *If*

$$U(x) = \Phi\big(-2\log(-\log G(x))\big)$$

is a record value distribution, then for any given β, $-\infty < \beta < \infty$,

$$V(x) = \Phi\big(\beta - 2\log(-\log G(x))\big)$$

is a record value distribution belonging to the same class as $U(x)$.

In particular,
(a_1) *if* $U(x) = \Phi(x)$, *then* $V(x) = U(x+\beta)$;
(b_1) *if* $U(x) = \Phi_{1,\alpha}(x)$, *then* $V(x) = U(Ax)$, *where* $A = e^{\beta/\alpha}$,
(c_1) *if* $U(x) = \Phi_{2,\alpha}(x)$, *then* $V(x) = U(Ax)$, *where* $A = e^{-\beta/\alpha}$.

The following theorem elucidates the role of a-tail equivalence in the limit behaviour of record values.

THEOREM 4. *If continuous distribution functions F and G are such that their associated distribution functions are a-tail equivalent,* $0 < a < \infty$, *then*

$$F \in DR(U) \quad \textit{iff} \quad G \in DR(U),$$

where U is a record value distribution.

Proof. Let us assume that $F \in DR(U)$. Then, by Theorem 1,

$$F \in DR(U) \quad \text{iff} \quad H_F \in DM(V),$$

where $U = \Phi\big(-2\log(-\log V)\big)$. Hence, on the basis of a-tail equivalence of the associated distribution functions H_F and H_G, and Theorem 2, we obtain $H_G \in DM(V^{1/a})$. Hence, using Theorem 1 again, we have

$$G \in DR(U_1), \quad \text{where } U_1(x) = \Phi\Big(2\log a - 2\log\big(-\log V(x)\big)\Big).$$

Therefore, by Lemma 1, there exist $A > 0$ and B such that

$$G(x) \in DR\big(U(Ax+B)\big).$$

Now it is easy to see that if $G(x) \in DR\big(U(Ax+B)\big)$ with normalizing constants $a_n > 0$ and b_n, $n \geqslant 1$, then

$$G \in DR(U) \quad \text{with } a_n' = \frac{1}{A} a_n,\ b_n' = b_n - \frac{B}{A} a_n.$$

Similarly, one can prove that $G \in DR(U)$ implies $F \in DR(U)$.

For evaluation purposes it is convenient to express the conditions of Theorem 4 in terms of hazard functions corresponding to F and G instead of their associated distribution functions.

Definition. The distribution functions F and G are *β-hazard equivalent* if $x_0^F = x_0^G = x_0$, $x_0 \leqslant \infty$, and

$$\lim_{x \to x_0 -} \big(\sqrt{R_F(x)} - \sqrt{R_G(x)}\big) = \beta, \quad -\infty < \beta < \infty. \tag{10}$$

It is easy to see that F and G being a-tail equivalent are 0-*hazard equivalent.*

LEMMA 2. *The distribution functions F and G are β-hazard equivalent,* $-\infty < \beta < \infty$, *if and only if their associated distributions are $e^{-\beta}$-tail equivalent.*

Proof. The statement follows from the equalities

$$x_0^{H_F} = x_0^F, \quad x_0^{H_G} = x_0^G,$$

and

$$\frac{1-H_F(x)}{1-H_G(x)} = \exp\{-(\sqrt{R_F(x)}-\sqrt{R_G(x)})\}.$$

By Lemma 2 and Theorem 4 we have the following result.

THEOREM 4'. *If continuous distribution functions F and G are β-hazard equivalent, $-\infty < \beta < \infty$, then*

$$F \in DR(U) \quad \textit{iff} \quad G \in DR(U),$$

where U is a record value distribution.

We observe that β-hazard equivalence, $-\infty < \beta < \infty$, does not imply, in general, tail equivalence with ratio a, $0 < a < \infty$.

Indeed, for instance, the distribution functions

$$F(x) = 1-\exp\{-(\tan x+\sin x)^2\}, \quad 0 \leqslant x < \pi/2,$$

and

$$G(x) = 1-\exp\{-(\tan x)^2\}, \quad 0 \leqslant x < \pi/2,$$

are 1-hazard equivalent but not tail equivalent, as

$$\lim_{x\to\pi/2-} \frac{1-F(x)}{1-G(x)} = 0.$$

On the basis of the remark before Lemma 2 and Theorem 4' we get the following obvious assertion:

THEOREM 5. *If F and G are continuous and a-tail equivalent distribution functions, $0 < a < \infty$, then*

$$F \in DR(U) \quad \textit{iff} \quad G \in DR(U),$$

where U is a record value distribution.

THEOREM 6. *Let F and G be continuous distribution functions such that*

(11) $F \in DR(U)$ *with normalizing constants* $a_n > 0$ *and* b_n, $n \geqslant 1$,

(12) $G \in DR(V)$ *with the same constants,*

where U and V are record value distributions.

Then the distribution functions F and G are β-hazard equivalent, $-\infty < \beta < \infty$, and there exist constants $A > 0$ and B such that $V(x) = U(Ax+B)$, where

(a_2) *if* $U(x) = \Phi(x)$, *then* $x_0 \leqslant \infty$, $A = 1$, *and* $B = -2\beta$;

(b_2) *if* $U(x) = \Phi_{1,a}(x)$, *then* $x_0 = \infty$, $A = \exp\{-2\beta/a\}$, *and* $B = 0$;

(c_2) *if* $U(x) = \Phi_{2,a}(x)$, *then* $x_0 < \infty$, $A = \exp\{2\beta/a\}$, *and* $B = 0$.

Proof. We know that

$$U = \Phi\big(-2\log(-\log U_1)\big) \quad \text{and} \quad V = \Phi\big(-2\log(-\log V_1)\big),$$

where U_1 and V_1 are extreme value distributions, respectively. The normalizing constants a_n and b_n can be replaced by functions $a(s) > 0$ and $b(s)$, $s \in (0, \infty)$ (see [2]). Since (11) and (12) imply

$$\lim_{n\to\infty} H_F^n(a_n' x + b_n') = U_1(x), \tag{13}$$

$$\lim_{n\to\infty} H_G^n(a_n' x + b_n') = V_1(x), \tag{14}$$

where

$$a_n' = a\big((\log n)^2\big) \quad \text{and} \quad b_n' = b\big((\log n)^2\big)$$

(see [2]), we can use Theorem 3. Therefore, H_F and H_G are γ-tail equivalent, $0 < \gamma < \infty$, and, by Lemma 2, F and G are β-hazard equivalent, $\beta = -\log\gamma$, which proves the first statement of Theorem 6.

Now we prove (b_2). If $U(x) = \Phi_{1,a}(x)$, then $U_1(x) = \Phi_{a/2}(x)$, whence, by (b), $x_0 = \infty$, $B = 0$, and $V_1(x) = \Phi_{a/2}(\gamma^{2/a}x)$. Therefore,

$$V(x) = \Phi\Big(-2\log\big(-\log\exp\{-(\gamma^{2/a}x)^{-a/2}\}\big)\Big) = \Phi_{1,a}(Ax),$$

where $A = \gamma^{2/a} = \exp\{-2\beta/a\}$, which proves ($b_2$).

The proof of (a_2) and (c_2) is analogous.

We shall characterize closer the statement of Theorem 4′ (or, equivalently, Theorem 4). Theorem 4′ does not give precisely the normalizing constants for which the equivalence

$$F \in DR(U) \quad \text{iff} \quad G \in DR(U)$$

holds.

We observe that, under the assumptions of Theorem 4′, there exist $A > 0$ and B such that

$$F(x) \in DR\big(U(x)\big) \quad \text{and} \quad G(x) \in DR\big(U(Ax+B)\big)$$

with the same normalizing constants. The constants A and B depend on β and on the type of the record value distribution U.

THEOREM 7. *Let F and G be continuous β-hazard equivalent distribution functions, $-\infty < \beta < \infty$. If*

(15) $F(x) \in DR\big(U(x)\big)$ *with normalizing constants* $a_n > 0$ *and* b_n, $n \geqslant 1$,

then constants $A > 0$ *and* B *given by* (a_2), (b_2) *or* (c_2) *are such that*

(16) $G(x) \in DR\big(U(Ax+B)\big)$ *with normalizing constants* a_n *and* b_n.

Proof. It is known (see [3]) that (15) holds iff

$$\lim_{n\to\infty} \exp\{\sqrt{n}\}\big(1-H_F(a_nx+b_n)\big) = \exp\left\{-\frac{1}{2}g(x)\right\}, \tag{17}$$

where $g(x) = \Phi^{-1}\big(U(x)\big)$. From β-hazard equivalence, by Lemma 2, we infer that

$$\lim_{x\to x_0-} \frac{1-H_F(x)}{1-H_G(x)} = e^{-\beta}. \tag{18}$$

It is easy to see that for all x such that $|g(x)| < \infty$ equality (17) implies

$$a_nx+b_n \to x_0- \quad \text{as } n\to\infty.$$

Hence, for $|g(x)| < \infty$ (17) and (18) give

$$\begin{aligned} &\lim_{n\to\infty} \exp\{\sqrt{n}\}\big(1-H_G(a_nx+b_n)\big) \\ &= \lim_{n\to\infty} \frac{\exp\{\sqrt{n}\}\big(1-H_G(a_nx+b_n)\big)}{\exp\{\sqrt{n}\}\big(1-H_F(a_nx+b_n)\big)} \exp\{\sqrt{n}\}\big(1-H_F(a_nx+b_n)\big) \\ &= \exp\{\beta\}\exp\left\{-\frac{1}{2}g(x)\right\} = \exp\left\{-\frac{1}{2}\big(g(x)-2\beta\big)\right\}. \end{aligned} \tag{19}$$

Since g and H_G are continuous non-decreasing functions, we see that (19) holds for every x. But (19) holds iff $G \in DR(V)$ with normalizing constants a_n and b_n, where

$$V(x) = \Phi\big(g(x)-2\beta\big) \quad \text{and} \quad g(x) = \Phi^{-1}\big(U(x)\big).$$

Thus, by Lemma 1, V and U belong to the same class. Therefore, there exist $A > 0$ and B depending on β and $g(x)$ such that $V(x) = U(Ax+B)$ Putting instead of $g(x)$ the functions x, $\log x^a$, and $\log(-x)^{-a}$, we see that A and B are given by (a_2), (b_2) and (c_2), respectively.

Resnick stated (see [3]) that it is difficult to find functions, except $R(x) = (\frac{1}{2}a\log x)^2$, $x \geqslant 1$, which are attracted to $\Phi_{1,a}(x)$. Theorem 4′ allows us to construct other distribution functions belonging to the domain of attraction of $\Phi_{1,a}$.

THEOREM 8. *Let F be a continuous distribution function with the right end x_0, $x_0 \leqslant \infty$. Suppose that $f\colon \mathbf{R}\to\mathbf{R}$ and $g\colon \mathbf{R}\to\mathbf{R}$ are continuous non-decreasing functions such that*

$$\lim_{x\to-\infty} f(x) = 0, \quad \lim_{x\to x_0-} f(x) = \beta,\ 0 \leqslant \beta < \infty,$$

$$\liminf_{x\to-\infty} g(x) > -\infty, \quad \lim_{x\to x_0-} \big(1-g(x)\big)\sqrt{R_F(x)} = a,\ 0 \leqslant a < \infty.$$

If $R_F \in DR(U)$, where U is a record value distribution, then the function

$$R_1(x) = \big(g(x)\sqrt{R_F(x)}+f(x)\big)^2 \tag{20}$$

is such that $R_1 \in DR(U)$.

Proof. It is easy to verify that

$$F_1(x) = 1-\exp\{-R_1(x)\}$$

is a continuous distribution function with the right end x_0. The distribution functions F and F_1 are $(a-\beta)$-hazard equivalent. Indeed,

$$\lim_{x\to x_0-}\big(\sqrt{R_F(x)}-\sqrt{R_G(x)}\big) = \lim_{x\to x_0-}\Big(\big(1-g(x)\big)\sqrt{R_F(x)}-f(x)\Big) = a-\beta.$$

Therefore, by Theorem 4′, $R_1 \in DR(U)$, which completes the proof.

Example. Let

$$g(x) \equiv 1 \quad \text{and} \quad f(x) = (\pi/2+\arctan x)^c,$$

where c is a positive constant. Then

$$R_1(x) = \begin{cases} (\pi/2+\arctan x)^{2c} & \text{if } x<1, \\ \big(\tfrac{1}{2}a\log x+(\pi/2+\arctan x)^c\big)^2 & \text{if } x\geqslant 1 \end{cases}$$

fulfils the assumptions of Theorem 8. Since

$$R(x) = \left(\frac{1}{2}\,a\log x\right)^2 \in DR\big(\Phi_{1,a}(x)\big)$$

(see [2]), we have

$$R_1(x) \in DR\big(\Phi_{1,a}(x)\big).$$

Remark. It is easy to see that the assumptions of Theorem 8 can be weakened. It is enough to take f and g such that R_1 defined by (20) is the hazard function corresponding to a continuous distribution function and

$$\lim_{x\to x_0-}\Big(\big(1-g(x)\big)\sqrt{R_F(x)}-f(x)\Big) = \beta, \qquad -\infty<\beta<\infty,$$

holds.

3. Concluding observations. Theorem 6 states that if $F \in DR(U)$ and $G \in DR(U)$ with the same normalizing constants, then F and G are hazard equivalent. The assumption that the above-mentioned convergence holds with the same normalizing constants is necessary. This is a consequence of the following theorem:

THEOREM 9. *Let F be a continuous distribution function and let $G(x) = 1-\big(1-F(x)\big)^c$, where c is a positive constant. Then*

$$F \in DR(U) \tag{21}$$

iff

$$G \in DR(V), \quad \textit{where } V(x) = \Phi\big(\sqrt{c}\,\Phi^{-1}\big(U(x)\big)\big). \tag{22}$$

Proof. It was proved in [2] (Theorem 3.1) that $F \in DR(U)$ with normalizing constants $a_n > 0$ and b_n, $n \geqslant 1$, iff

$$\lim_{n\to\infty} \frac{R_F(a_n x + b_n) - n}{\sqrt{n}} = g(x), \tag{23}$$

where $g(x) = \Phi^{-1}\big(U(x)\big)$. By switching from normalizing constants a_n and b_n, $n \geqslant 1$, to functions $a(s) > 0$ and $b(s)$ (see [3]), equality (23) can be replaced by

$$\lim_{s\to\infty} \frac{R_F\big(a(s)x + b(s)\big) - s}{\sqrt{s}} = g(x). \tag{24}$$

Let us set $y = cs$. Then, by (24),

$$\lim_{y\to\infty} \frac{cR_F\big(a(y/c)x + b(y/c)\big) - y}{\sqrt{y}} = \sqrt{c}\,g(x).$$

Putting $a_n' = a(n/c)$ and $b_n' = b(n/c)$, we obtain

$$\lim_{n\to\infty} \frac{cR_F(a_n x + b_n) - n}{\sqrt{n}} = \sqrt{c}\,g(x). \tag{25}$$

Let us observe that $R_G(x) = cR_F(x)$. Therefore, by Theorem 3.1 from [3], we see that condition (25) is equivalent to (22). This completes the proof.

Let now $F(x) \in DR\big(\Phi(x)\big)$. Then, by Theorem 9,

$$G(x) = 1 - \big(1 - F(x)\big)^c \in DR\big(\Phi(\sqrt{c}\,x)\big).$$

It is obvious that also $G(x) \in DR\big(\Phi(x)\big)$. For $c \neq 1$, F and G are not β-hazard equivalent since

$$\lim_{x\to x_0-} \big(\sqrt{R_F(x)} - \sqrt{R_G(x)}\big) = \lim_{x\to x_0-} (1 - \sqrt{c})\sqrt{R_F(x)} = \pm\infty.$$

Thus we have shown that the hazard equivalence of two continuous distribution functions F and G is not a necessary conditions for the equivalence

$$F \in DR(U) \quad \text{iff} \quad G \in DR(U).$$

From Theorem 9 we obtain another interesting property of limit behaviour of record values.

Let

$$Y_n = \min\{X_{(n-1)k+1}, \ldots, X_{nk}\}, \quad n \geqslant 1,$$

where k is a fixed positive integer. If $\{X_n, n \geqslant 1\}$ is a sequence of independent, identically distributed random variables with a common continuous distribution function F, then, by Theorem 9,

$$F \in DR(U) \quad \text{iff} \quad G \in DR(V),$$

where $G(x) = \mathrm{P}[Y_i \leqslant x]$ and $V(x) = \Phi\big(\sqrt{k}\,\Phi^{-1}(U(x))\big)$.

Since $\sqrt{k}$ appears only as a location or scale parameter, U and V belong to the same type of record value distributions (if $F \in DR(\Phi)$, then $G \in DR(\Phi)$; if $F \in DR(\Phi_{1,a})$, then $G \in DR(\Phi_{1,\sqrt{k}a})$; if $F \in DR(\Phi_{2,a})$, then $G \in DR(\Phi_{2,\sqrt{k}a})$).

Finally, as an application of Theorem 4′ (and also Theorem 5), we prove a result which is similar to that for the maximum of a random number of random variables [2].

THEOREM 10. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables with a common continuous distribution function F, and the right end x_0, $x_0 \leqslant \infty$. Let $\{N_n, n \geqslant 1\}$ be a sequence of independent, identically distributed random variables with*

$$\mathrm{P}[N_i = k] = p_k, \quad k \geqslant 1, \ \sum_{k=1}^{\infty} p_k = 1, \mathrm{E}N_i = \sum_{k=1}^{\infty} kp_k < \infty.$$

Suppose that $\{N_n, n \geqslant 1\}$ and $\{X_n, n \geqslant 1\}$ are independent of each other and

$$S_n = \sum_{i=1}^{n} N_i, \quad Y_n = \max\{X_{S_{n-1}+1}, \ldots, X_{S_n}\}.$$

Then

$$F \in DR(U) \quad \textit{iff} \quad F_{Y_1} \in DR(U).$$

Proof. It is easy to see that the Y_n's are independent, identically distributed random variables with a common continuous distribution function

$$F_{Y_1}(x) = \sum_{k=1}^{\infty} F^k(x) p_k.$$

In [3] it has been proved that

$$\lim_{x \to x_0 -} \frac{1 - F_{Y_1}(x)}{1 - F(x)} = \mathrm{E}N_1.$$

But, $0 < \mathrm{E}N_1 < \infty$, therefore, by Theorem 5, we get the required statement.

REFERENCES

[1] B. V. Gnedenko, *Sur la distribution limite du terme maximum d'une série aléatoire*, Annals of Mathematics 44 (1943), p. 423-453.

[2] S. I. Resnick, *Tail equivalence and its applications*, Journal of Applied Probability 8 (1971), p. 135-156.

[3] — *Limit laws for record values*, Stochastic Processes and their Applications 1 (1973), p. 67-82.

Reçu par la Rédaction le 12. 7. 1975