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ON THE FIRST CHERN CLASS OF A COMPLEX SUBMANIFOLD IN AN ALMOST HERMITIAN MANIFOLD AND THE NORMAL CONNECTION

BY

MANUEL BARROS AND FLORENTINO G. SANTOS (GRANADA)

1. Introduction. In this note we shall study the first Chern class of the normal bundle of a σ-submanifold M in an almost Hermitian manifold whose normal curvature tensor satisfies the condition

$$\text{(A)} \qquad R^D(X, Y)\xi = fg(X, JY)J\xi$$

for any vector fields X and Y tangent to M and ξ normal to M, where f is a differentiable function on M.

Condition (A) has been introduced by Ishihara in [6]. In his paper he classifies the complex submanifolds in the complex projective space satisfying Condition (A). Therefore, our main result can be considered as a topological interpretation of Condition (A).

We shall give some examples of σ-submanifolds which satisfy Condition (A).

2. Some lemmas. In this section we give some lemmas about Condition (A). Therefore, we assume that $\bar{M}$ is an $(n+p)$-dimensional almost Hermitian manifold, and M an n-dimensional σ-submanifold which satisfies Condition (A). We shall denote by $X, Y, Z, \ldots$ (respectively, $\xi, \eta, \ldots$) the vector fields tangent to M (respectively, normal to M). Then we have the following lemmas.

LEMMA 1. *If $\bar{M}$ is a QK-manifold, then f is a constant function on M.*

Proof. It is clear that we have the following Bianchi's identity:

$$(2.1) \qquad \underset{X,Y,Z}{\sigma} (\tilde{\nabla}_X R^D)(Y, Z) = 0,$$

where $\tilde{\nabla}$ denotes the van der Waerden–Bortolotti covariant derivative of R^D given by

$$(2.2) \qquad (\tilde{\nabla}_X R^D)(Y, Z)\xi = D_X R^D(Y, Z)\xi - R^D(\nabla_X Y, Z)\xi - R^D(Y, \nabla_X Z)\xi - R^D(Y, Z)D_X\xi.$$

As M is a σ-submanifold, we have

$$D_X J\xi = (\tilde{\nabla}_X J)\xi + JD_X\xi. \tag{2.3}$$

We compute the covariant derivative of (A) and use (2.1)–(2.3). Hence

$$(Xf)g(Y, JZ)J\xi + (Zf)g(X, JY)J\xi + (Yf)g(Z, JX)J\xi$$
$$+f\{g((\nabla_X J)Z, Y)J\xi + g((\nabla_Y J)X, Z)J\xi + g((\nabla_Z J)Y, X)J\xi\}$$
$$+f\{g(Y, JZ)(\bar{\nabla}_X J)\xi + g(Z, JX)(\bar{\nabla}_Y J)\xi + g(X, JY)(\bar{\nabla}_Z J)\xi\} = 0.$$

Now we choose $Z = JY$ and $g(X, Y) = g(X, JY) = 0$ and use the fact that $\bar{M}$ and M are QK-manifolds:

$$(Xf)g(Y, Y)J\xi + fg(Y, Y)(\bar{\nabla}_X J)\xi = 0.$$

Then

$$(Xf)J\xi + f(\bar{\nabla}_X J)\xi = 0 \tag{2.4}$$

and

$$-(JXf)\xi + f(\bar{\nabla}_{JX} J)J\xi = 0. \tag{2.5}$$

As $\bar{M}$ is a QK-manifold, adding (2.4) and (2.5) we get

$$(Xf)J\xi - (JXf)\xi = 0,$$

which proves that f is a constant.

LEMMA 2. *Let $\bar{M}$ be an almost Hermitian manifold, and M a σ-submanifold of $\bar{M}$ which satisfies Condition* (A). *Then the *-Ricci tensors S^* and $\bar{S}^*$ of M and $\bar{M}$, respectively, are related by*

$$\bar{S}^*(X, Y) = S^*(X, Y) + pfg(X, Y)$$

for any X and Y.

Proof. Using the Gauss equation and the definition of σ-submanifold we have

$$\bar{S}^*(X, Y) = S^*(X, Y) + 2\sum_{i=1}^{n} g(\sigma(X, E_i), \sigma(Y, E_i)) + \sum_{\alpha=1}^{p} \bar{R}(X, JY, J\xi_\alpha, \xi_\alpha), \tag{2.6}$$

where $\{E_i, JE_i, \xi_\alpha, J\xi_\alpha\}$, $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leqslant p$, is an orthonormal local frame field on $\bar{M}$ with $\{E_i, JE_i\}$, $1 \leqslant i \leqslant n$, a local frame field on M and $\{\xi_\alpha, J\xi_\alpha\}$, $1 \leqslant \alpha \leqslant p$, a local frame of normal sections on M.

By Ricci's equation and Condition (A),

$$\bar{R}(X, JY, J\xi_\alpha, \xi_\alpha) = fg(X, Y) - 2g(A^2_{\xi_\alpha}X, Y). \tag{2.7}$$

On the other hand,

$$\sum_{i=1}^{n} g(\sigma(X, E_i), \sigma(Y, E_i)) = \sum_{\alpha=1}^{p} g(A^2_{\xi_\alpha} X, Y). \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) we find the required result.

Now we shall prove that the converse of Lemma 2 holds if the complex codimension $p = 1$.

LEMMA 3. *Let M be a σ-hypersurface of an almost Hermitian manifold $\bar{M}$. Then M satisfies Condition* (A) *if and only if*

$$\bar{S}^*(X, Y) = S^*(X, Y) + fg(X, Y)$$

for any X and Y.

Proof. The necessary condition is proved in Lemma 2.

If $\bar{S}^*(X, Y) = S^*(X, Y)+fg(X, Y)$, then from (2.6) and (2.8) we obtain

$$fg(X, Y) = 2g(A^2_\xi X, Y)+\bar{R}(X, JY, J\xi, \xi).$$

Hence, by Ricci's equation, this implies that

$$R^D(X, JY, J\xi, \xi) = fg(X, Y),$$

i.e., the normal connection of M satisfies Condition (A).

Remarks. (1) If $\bar{M}$ is an F-space, that is, an almost Hermitian manifold satisfying the Kaehler identity, in particular a Kaehler manifold, then $\bar{S}^* = \bar{S}$, and Lemma 2 has been proved in [6].

(2) The case $f = 0$, that is, the normal connection D is flat, has been studied in [3].

3. Some examples. In this section we give some examples of complex submanifolds satisfying Condition (A).

(1) Suppose that $\bar{M}(c)$ is a complex-space-form of constant holomorphic sectional curvature c, and M is a totally geodesic complex submanifold. Then M satisfies Condition (A) with $f = c/2$.

(2) Any Einsteinian complex hypersurface of a complex-space-form satisfies Condition (A).

(3) Let $G = U(p+q+r)$ be the unitary group of order $n = p+q+r$. Then $H = U(p) \times U(q) \times U(r)$ is a closed subgroup of G and $\bar{M} = G/H$ is a reductive homogeneous space. If $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, respectively, then the tangent space to $\bar{M}$ at a point is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A_{12} & A_{13} \\ -\bar{A}^t_{12} & 0 & A_{23} \\ -\bar{A}^t_{13} & -\bar{A}^t_{23} & 0 \end{pmatrix} \;\middle|\; A_{ij} \in \mathcal{M}_{ij}(C) \right\},$$

where $\mathcal{M}_{ij}(C)$ is the set of complex matrices of order $i \times j$.

On $\bar{M}$ we consider the metric obtained by the projection of the only bi-invariant metric on G. We define

$$\mathfrak{m}_{12} = \left\{ \begin{pmatrix} 0 & A_{12} & 0 \\ -\bar{A}^t_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid A_{12} \in \mathscr{M}_{12}(C) \right\};$$

$\mathfrak{m}_{13}$ and $\mathfrak{m}_{23}$ are defined in an analogous way. So, we have

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \qquad \mathfrak{m} = \mathfrak{m}_{12} \oplus \mathfrak{m}_{13} \oplus \mathfrak{m}_{23}.$$

The Riemannian connection on $\bar{M}$ is given by

$$\bar{\nabla}_X Y = \tfrac{1}{2}[X, Y]_{\mathfrak{m}}.$$

The linear mapping J_{pq}: $\mathfrak{m}_{pq} \to \mathfrak{m}_{pq}$ defined by $J_{pq}(A_{pq}) = (iA_{pq})$ is obviously an automorphism of $\mathfrak{m}_{pq}$ which satisfies $J^2_{pq} = -I$. Let us extend J_{pq} in $\mathfrak{m}$ as

$$J = \alpha_{12} J_{12} + \alpha_{13} J_{13} + \alpha_{23} J_{23}.$$

Then J is an almost complex structure if and only if $\alpha^2_{ij} = 1$. Moreover, these almost complex structures are almost Hermitian structures. The following facts are proved in [2]. We denote by $\mathscr{J} = \{J(\alpha_{ij})\}$ the set of almost Hermitian structures which have been obtained before:

(1) $\mathscr{J}$ does not contain any Kaehlerian structure.

(2) The structures $J = J_{12} - J_{13} + J_{23}$ and $J = -J_{12} + J_{13} - J_{23}$ are nearly-Kaehlerian.

(3) All the structures of $\mathscr{J}$ which are not nearly-Kaehlerian are Hermitian.

Moreover, as a particular case of the Theorem obtained in [2], we infer that the complex projective space CP^1 can be isometrically immersed in

$$\bar{M} = U(3)/U(1) \times U(1) \times U(1)$$

as a complex totally geodesic submanifold.

Now a simple computation proves the following

PROPOSITION 1. *CP^1 is a complex submanifold of* $(\bar{M}, J = J_{12} - J_{13} + J_{23})$ *which satisfies Condition* (A) *with* $f = -3/4$.

PROPOSITION 2. *CP^1 is a complex submanifold of* $(\bar{M}, J = J_{12} + J_{13} - J_{23})$ *which satisfies Condition* (A) *with* $f = 3/4$.

4. A topological interpretation of Condition (A). Let $\bar{M}$ be an $(n+p)$-dimensional QK-manifold, and M an n-dimensional σ-submanifold satisfying Condition (A). So, we define on M a 2-form Φ' by

$$\Phi'(X, Y) = pf\Phi(X, Y) - \frac{1}{2} \sum_{\alpha=1}^{p} g((\bar{\nabla}_{JX} J)\xi_\alpha, (\bar{\nabla}_Y J)\xi_\alpha) \tag{4.1}$$

for any X and Y, where Φ is the fundamental 2-form on M and $\{\xi_\alpha, J\xi_\alpha\}$, $1 \leqslant \alpha \leqslant p$, is an orthonormal local frame of normal sections on M.

It is obvious that Φ' is globally well-defined and J-invariant. If $\bar{M}$ is a Kaehlerian manifold, then Φ' is proportional to the fundamental 2-form Φ.

The following Theorem gives a topological interpretation of Condition (A):

THEOREM. *Let M be a σ-submanifold of a QK_3-manifold $\bar{M}$. Assume that M satisfies Condition* (A). *Then Φ' is a closed 2-form and*

$$c_1(T^\perp M) = \frac{1}{2\pi}[\Phi'].$$

Proof. It is easy to see that the first Chern forms of $\bar{M}$ and M are given, respectively, by

$$\bar{\gamma}_1(\bar{X}, J\bar{Y}) = \frac{1}{2\pi}\Big\{\bar{S}^*(\bar{X}, \bar{Y}) - \frac{1}{2}\sum_{i=1}^{n} g\big((\bar{\nabla}_{\bar{X}} J)E_i, (\bar{\nabla}_{\bar{Y}} J)E_i\big) - \frac{1}{2}\sum_{\alpha=1}^{p} g\big((\bar{\nabla}_{\bar{X}} J)\xi_\alpha, (\bar{\nabla}_{\bar{Y}} J)\xi_\alpha\big)\Big\}, \tag{4.2}$$

$$\gamma_1(X, JY) = \frac{1}{2\pi}\Big\{S^*(X, Y) - \frac{1}{2}\sum_{i=1}^{n} g\big((\nabla_X J)E_i, (\nabla_Y J)E_i\big)\Big\} \tag{4.3}$$

for any X, Y and any $\bar{X}$, $\bar{Y}$ tangent to $\bar{M}$, where $\{E_i, JE_i, \xi_\alpha, J\xi_\alpha\}$, $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leqslant p$, is an orthonormal local frame field on $\bar{M}$ with $\{E_i, JE_i\}$, $1 \leqslant i \leqslant n$, a local frame field on M and $\{\xi_\alpha, J\xi_\alpha\}$, $1 \leqslant \alpha \leqslant p$, a local frame of normal sections on M.

Since M is a σ-submanifold, it is easy to see that $(\bar{\nabla}_X J)Y = (\nabla_X J)Y$. Thus from Lemma 2, (4.2), and (4.3) we get

$$\bar{\gamma}_1(X, JY) = \gamma_1(X, JY) = \frac{1}{2\pi}\Big\{pfg(X, Y) - \frac{1}{2}\sum_{\alpha=1}^{p} g\big((\bar{\nabla}_X J)\xi_\alpha, (\bar{\nabla}_Y J)\xi_\alpha\big)\Big\}. \tag{4.4}$$

Since $\bar{M}$ and M are QK_3-manifolds, the first Chern forms $\bar{\gamma}_1$ and γ_1 are J-invariant. Then from (4.4) we obtain

$$\bar{\gamma}_1(X, Y) = \gamma_1(X, Y) + \frac{1}{2\pi}\Big\{pfg(JX, Y) - \frac{1}{2}\sum_{\alpha=1}^{p} g\big((\bar{\nabla}_{JX} J)\xi_\alpha, (\bar{\nabla}_Y J)\xi_\alpha\big)\Big\}$$

and from (4.1) we get

$$\bar{\gamma}_1(X, Y) = \gamma_1(X, Y) + \frac{1}{2\pi}\Phi'(X, Y).$$

This proves that $d\Phi' = 0$; moreover,

$$c_1(T\bar{M}/M) = c_1(TM) + \frac{1}{2\pi}[\Phi']. \tag{4.5}$$

On the other hand, $T\bar{M}/M = TM \oplus T^{\perp}M$. Thus

$$c_1(T\bar{M}/M) = c_1(TM) + c_1(T^{\perp}M). \tag{4.6}$$

Now, (4.5) and (4.6) give the result.

Remarks. (1) If $\bar{M}$ is a Kaehlerian manifold, $\Phi' = pf\Phi$. Therefore, if M is a complex submanifold of $\bar{M}$ satisfying Condition (A), then M is cohomologically Einstein relatively to the normal bundle [7].

(2) Moreover, if M has a flat normal connection, then $c_1(T^{\perp}M) = 0$. This result has been proved in [3].

COROLLARY 1. *Let M be a compact σ-submanifold of a QK_3-manifold $\bar{M}$ which has a flat normal connection ($f = 0$). Then the Chern number $(-1)^n C_1^n(T^{\perp}M)$ of $T^{\perp}M$ is nonnegative.*

Proof. By (4.1), $\Phi'(X, JX) \leqslant 0$ for any X. We choose a basis $\{X_i, JX_i\}$, $1 \leqslant i \leqslant n$, which diagonalizes the J-invariant symmetric operator $\delta(X, Y) = \Phi'(X, JY)$.

Then, from the Theorem we obtain

$$(-1)^n C_1^n(T^{\perp}M) = (-1)^n \int_M (\Phi' \wedge \ldots \wedge \Phi')(X_1, JX_1, \ldots, X_n, JX_n)\,\omega,$$

where ω is the volume element on M such that

$$\omega(X_1, JX_1, \ldots, X_n, JX_n) = 1.$$

This proves the corollary.

It is clear that $\Phi' = 0$ if and only if it is possible to choose a local frame of normal vector fields $\{\xi_\alpha, J\xi_\alpha\}$, $1 \leqslant \alpha \leqslant p$, which are parallel relative to the normal connection, that is, $D\xi_\alpha = DJ\xi_\alpha = 0$, $1 \leqslant \alpha \leqslant p$. In this case, if $\bar{M}$ is a nearly-Kaehlerian manifold and M is a complex submanifold with flat normal connection, we obtain $\bar{S}(X, Y) = S(X, Y)$ for any X and Y. Moreover, if M is a complex hypersurface of $\bar{M}$ with flat normal connection, then $\Phi' = 0$ if and only if $\bar{S}(X, Y) = S(X, Y)$. Consequently, we obtain

COROLLARY 2. *Let $\bar{M}$ be an Einstein almost Tachibana space, and M a complex submanifold of $\bar{M}$ with flat normal connection. If $\Phi' = 0$, then M is also an Einstein almost Tachibana space. Moreover, $\bar{M}$ and M have the same scalar curvature.*

COROLLARY 3. *Let M be a complex hypersurface of an almost Tachibana space $\bar{M}$ with flat normal connection. If both M and $\bar{M}$ are Einstein and have the same scalar curvature, then $\Phi' = 0$.*

REFERENCES

[1] M. Barros and F. G. Santos, *On normal curvature tensor of Kaehler submanifolds*, Chinese J. Math. 8 (1980), pp. 105–112.

[2] M. Barros and F. Urbano, *Differential geometry of* $U(p+q+r)/U(p)\times U(q)\times U(r)$, Rend. Circ. Mat. Palermo 30 (1981), pp. 83–96.

[3] B. Y. Chen and H. S. Lue, *On normal connection of Kaehler submanifolds*, J. Math. Soc. Japan 27 (1975), pp. 550–556.

[4] A. Gray, *Nearly-Kaehler manifolds*, J. Differential Geometry 4 (1970), pp. 283–309.

[5] – M. Barros, A. M. Naveira and L. Vanhecke, *The Chern numbers of holomorphic vector bundles and formally holomorphic connections of complex vector bundles over almost complex manifolds*, J. Reine Angew. Math. 314 (1980), pp. 84–98.

[6] I. Ishihara, *Kaehler submanifolds satisfying a certain condition on normal connection*, Lincei Rend. Sc. fis. mat. e nat. 62 (1977), pp. 30–35.

[7] K. Ogiue, *Generalized scalar curvatures of cohomological Einstein Kaehler manifolds*, J. Differential Geometry 10 (1975), pp. 201–205.

DEPARTAMENTO DE GEOMETRÍA
FACULTAD DE CIENCIAS
UNIVERSIDAD DE GRANADA
GRANADA

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