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TWO EXAMPLES OF NON-SEPARABLE METRIZABLE SPACES

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In this note we shall give two examples, the first of which is an answer to a question of A. Pełczyński and the second to a question communicated to the author by P. Minc.

In our constructions the following observation due to A. H. Stone plays a fundamental role:

There is a zero-dimensional, complete, metrizable space T of weight $\aleph_1$ containing a subspace S such that

(1) *each separable subspace of S is countable,*

(2) *S is not a Borel set at any point of T, i.e., for every open non-empty subset U of T, $U \cap S$ is not a Borel set in T.*

For the proof let us take the subspace E of the Baire space $B(\aleph_1)$ (i.e. of the countable product of discrete spaces of cardinality $\aleph_1$) with the property that each separable subspace of E is countable, but E is not σ-discrete (see [5], 5.1). Let S be the non-locally σ-discrete kernel of E (see [4], Theorem 1) and let T be the closure of S in $B(\aleph_1)$. Then S is non-empty (since, by [4], Theorem 4, the space $E \setminus S$ is σ-discrete) and open non-empty subspaces of S are not σ-discrete. By [5], Theorem 2, no open non-empty subset of S is an absolutely Borel set, and hence (2) holds.

Example 1. *There exists a non-Borel subspace X of the Hilbert space H of weight $\aleph_1$ such that every separable subspace of X is contained in a closed subspace of X homeomorphic to the separable Hilbert space.*

We can assume that $S \subset H$. Take $X = H \setminus S$. Since, by (2), S is not an absolutely Borel space, X is not a Borel set in H. For every separable subspace A of X there exists a closed, linear, separable subspace H' of H containing A. The intersection $H' \cap S$ is separable, and hence, by (1), countable. Thus $H' \cap X = H' \setminus (H' \cap S)$ is the complement of a countable subset of the separable Hilbert space H' and, by a theorem of Anderson (see [1], Corollary 2), it is homeomorphic to the separable Hilbert space.

Example 2. *There exists a connected, metrizable space Y of weight $\aleph_1$, each separable subspace of which is zero-dimensional.*

Let $S(r)$ be the sphere with center 0 and radius r in the Hilbert space H of weight $\aleph_1$. Let $\overline{xy}$ denote the segment joining points $x, y \in H$, let R denote the real numbers, and let P and Q be the sets of irrational and rational numbers, respectively, of the open interval $(0, 1)$. For an arbitrary pair of disjoint subsets A, B of the unit sphere $S(1)$ we define the space $M(A, B)$ — the Knaster-Kuratowski broom over A and B (cf. [3], Section 46, II, Remark) — by setting

$$M(A, B) = \{tx\colon (x \in A \text{ and } t \in P) \text{ or } (x \in B \text{ and } t \in Q) \text{ or } (t = 0)\}.$$

We can assume that $T \subset S(1)$. Put

$$C = T \setminus S \quad \text{and} \quad Y = M(S, C).$$

We prove that Y is connected, repeating the argumentation of Knaster and Kuratowski [2], p. 241. Suppose that Y is not connected. Then there exists a closed subset L of H which is disjoint with Y and cuts Y. The set L cuts some segment $\overline{ao}$, where $a \in T$, and thus it cuts also every segment $\overline{xo}$ for x belonging to a sufficiently small neighbourhood V of a in T. For each $q \in Q$ let

$$L_q = \left\{\frac{1}{q}x\colon x \in S(q) \cap L\right\} \cap V.$$

We have $\bigcup_{q \in Q} L_q \subset V \cap S$, since in the opposite case it would be $x \in C \cap L_q$ for some $q \in Q$, but this implies $qx \in Y \cap L$, which is impossible. Suppose now that there exists

$$x \in (V \cap S) \setminus \bigcup_{q \in Q} L_q.$$

Since L cuts $\overline{xo}$, there is $t \in (0, 1)$ such that $tx \in L$. By our choice of x we have $t \in P$ and we get $tx \in Y \cap L$, which is a contradiction. We obtain

$$\bigcup_{q \in Q} L_q = V \cap S,$$

but this contradicts (2), as each L_q is a closed subset of V. Thus Y is connected.

Now let $A \subset Y$ be a separable space. There exists a separable space $T' \subset T$ such that $A \subset M(S', C')$ for $S' = S \cap T'$ and $C' = C \cap T'$. Since the space T' is zero-dimensional and separable, it can be embedded in R, and since, by (1), the set S' is countable, the space $M(S', C') \setminus \{0\}$ can be embedded in the subspace $Q \times P \cup P \times Q$ of the Euclidean plane, which is, as is easy to see, zero-dimensional. Thus A is zero-dimensional.

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