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A THEOREM ON ALMOST DISJOINT SETS

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THEOREM. *Let M be an infinite set of power a, f_i: $M \to M$ an injection for each $i < a$, $F_i = \{x \in M\colon f_i(x) = x\}$ and $F_{ij} = \{x \in M\colon f_i(x) = f_j(x)\}$. Then the following are equivalent*:

(i) *There exists a subset $X \subseteq M$ of power a such that any two members of the sequence*

$$X, f_0(X), f_1(X), \ldots, f_i(X), \ldots \quad (i < a)$$

are almost disjoint.

(ii) $|F_i| < a$, $|M - F_{ij}| = a$ *for all* $i, j < a$.

Proof. It is immediate that condition (ii) is necessary for (i) to hold so we shall prove sufficiency. We shall define the required set $X = \{x_k\colon k < a\}$ by induction on k. Specifically, let x_0 be an arbitrary member of M and suppose x_i for $i < k$ have all been defined. Let x_k be any member of M which satisfies

$$x_k \notin X_k \cup \bigcup_{s<k} f_s^{-1}(X_k) \cup \bigcup_{s<k} f_s(X_k) \cup \bigcup_{s,t<k} f_s^{-1} f_t(X_k),$$

where $X_k = \{x_i\colon i < k\}$.

We show first that

(1) $$f_s(x_k) \neq x_j \quad \text{for } j \neq k \text{ if } s \leqslant k.$$

Suppose $f_s(x_k) = x_j$ for $j < k$. Then, since $s \leqslant k \leqslant j$, the condition

$$x_j \notin \bigcup_{s \leqslant j} f_s(X_j)$$

applies, so we have $x_k \notin X_j$ and so $k \geqslant j$ contrary to our supposition. Now suppose $f_s(x_k) = x_j$ for $j < k$, i.e., $x_k = f_s^{-1}(x_j)$ and, therefore, since $s \leqslant k$, the condition

$$x_k \notin \bigcup_{s \leqslant k} f_s^{-1}(X_k)$$

applies, so that we have $x_j \notin X_k$ i.e. $j \geqslant k$ which is again contrary to our supposition. Thus (1) holds.

We now show that

(2) $$|f_s(X)\cap X|<a \quad \text{for all } s<a.$$

Suppose otherwise that for some $s<a$ we have $f_s(x_{k(i)}) = x_{j(i)}$ for all $i<a$. By (1) this means that $k(i) = j(i)$ for all $k(i),\ j(i) \geqslant s$ which contradicts $|F_s|<a$, and so (2) holds.

Finally we show that

(3) $$|f_s(X)\cap f_t(X)|<a \quad \text{for all } s, t<a.$$

Again we argue indirectly and suppose that for some $s, t<a$ (3) does not hold. Thus we suppose that

(4) $$f_s(x_{k(i)}) = f_t(x_{j(i)}) \quad \text{for all } i<a.$$

Put $i_0 = \max(s, t)$ and assume that $k(i), j(i) > i_0$. Then the condition $x_{k(i)} \notin f_s^{-1}f_t(X_{j(i)})$ holds so that (4) implies $x_{k(i)} \notin X_{j(i)}$, that is, $k(i) \geqslant j(i)$. Similarly the condition $x_{j(i)} \notin f_t^{-1}f_s(X_{k(i)})$ implies that $x_{j(i)} \notin X_{k(i)}$, that is, $j(i) \geqslant k(i)$. Thus (4) cannot hold if both $k(i), j(i) > i_0$ unless $k(i) = j(i)$ for all $i > i_0$ which contradicts the second condition of (ii) that $|M-F_{st}| = a$. On the other hand, if only $k(i) > i_0$ and $j(i) \leqslant i_0$ then (4) implies that f_s is not an injection, and similarly if $j(i) > i_0$ and $k(i) \leqslant i_0$ then (4) implies that f_t is not an injection. Thus (4) can only hold if both $k(i), j(i) \leqslant i_0$ which we assumed not to be the case and so the Theorem is proved.

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