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DERIVATES FOR SYMMETRIC FUNCTIONS

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A real-valued function f defined on $I_0 = [0,1]$ is said to be *symmetric* if, for each $x \in I_0^0 = (0,1)$,

$$f(x+h)+f(x-h)-2f(x) = o(1) \quad \text{as } h \to 0.$$

In [3], Neugebauer has studied the relation between continuity and symmetry and discovered properties that symmetric and continuous functions have in common. In particular, he has proved that if f is measurable and symmetric on I_0, then $\{x: \bar{f}^-(x) \neq \bar{f}^+(x) \text{ or } \underline{f}^-(x) \neq \underline{f}^+(x)\}$ is a set of the first category. This is an extension of a theorem obtained by him [2]. The purpose of this paper is to prove that the sets

$$\{x: \bar{f}_{\mathrm{ap}}^-(x) < \bar{f}^+(x) \text{ or } \bar{f}_{\mathrm{ap}}^+(x) < \bar{f}^-(x)\} \quad \text{and} \quad \{x: \bar{f}^s(x) \neq \bar{f}^+(x)\}$$

are of the first category if f is measurable and symmetric. It follows easily from the present work that

$$\bar{f}_{\mathrm{ap}}^s(x) \leqslant \bar{f}_{\mathrm{ap}}^+(x) = \bar{f}_{\mathrm{ap}}^-(x) = \bar{f}^+(x) = \bar{f}^-(x) = \bar{f}^s(x)$$

holds except possibly for a set of the first category. This observation for continuous functions has been noted by Evans and Humke [1]. Here, $\bar{f}_{\mathrm{ap}}^-(x)$, $\bar{f}_{\mathrm{ap}}^+(x)$, $\bar{f}_{\mathrm{ap}}^s(x)$, $\bar{f}^-(x)$, $\bar{f}^+(x)$, and $\bar{f}^s(x)$ denote the various upper derivates of f at x. For these definitions, we refer the reader to [5].

Throughout this paper, we assume that f is measurable, symmetric on I_0. For $A \subset I_0$, $|A|$ and $\bar{A}$ denote the Lebesgue outer measure and the closure of A, respectively. For real a,

$$aA = \{aa: a \in A\} \quad \text{and} \quad A-a = \{a-a: a \in A\}.$$

The symbol $\sim$ is used for set difference. Let G denote the set of points in I_0^0 at which f is continuous and let $H(x)$ for each $x \in I_0^0$ denote the set of positive reals h such that $x \pm h \in G$. It should be noted that G has full measure on I_0 and $I_0 \sim G$ is a set of the first category ([3], Theorems 1 and 4).

LEMMA 1. *If $a > 0$, $E \subset (0, a)$, and E has full measure in $(0, a)$, then*

(i) $$\bar{f}^+(x) = \limsup_{h\to 0,\ h\in E} \frac{f(x+h)-f(x)}{h},$$

(ii) $$\bar{f}^s(x) = \limsup_{h\to 0,\ h\in E} \frac{f(x+h)-f(x-h)}{2h}.$$

Proof. Here we give the proof for (i); (ii) can be proved similarly.

Let $x \in [0, 1)$ be fixed and let d_0 denote the limit on the right-hand side of (i). We need only to show that $\bar{f}^+(x) \leqslant d_0$ since the reverse inequality is obvious. Thus we may assume that $d_0 < +\infty$. For any $d > d_0$, there exists $\delta > 0$ such that

$$\frac{f(x+h)-f(x)}{h} < d \quad \text{for } h \in (0, \delta)\cap E.$$

Clearly, we may assume that $\delta \leqslant a$. From now on the proof parallels the one in [3], p. 30. For any $h \in (0, \delta)$, since E has full measure in $(0, a)$ $\supset (0, \delta)$, we can find a sequence $\{t_n\}$ decreasing to zero and $h \pm t_n \in (0, \delta)\cap E$ for every n. Thus for every n we have

$$\frac{f(x+h+t_n)-f(x)}{h+t_n} < d \quad \text{and} \quad \frac{f(x+h-t_n)-f(x)}{h-t_n} < d.$$

In case $d \geqslant 0$, we see that, for every n,

$$\frac{f(x+h-t_n)-f(x)}{h+t_n} = \frac{f(x+h-t_n)-f(x)}{h-t_n}\,\frac{h-t_n}{h+t_n} < d\,\frac{h-t_n}{h+t_n} < d,$$

and hence

$$\frac{f(x+h+t_n)+f(x+h-t_n)-2f(x)}{h+t_n} < 2d.$$

Let $n \to \infty$. Since f is symmetric, we obtain

$$\frac{2f(x+h)-2f(x)}{h} \leqslant 2d \quad \text{or} \quad \frac{f(x+h)-f(x)}{h} \leqslant d.$$

In case $d < 0$, we have, for every n,

$$\frac{f(x+h+t_n)-f(x)}{h-t_n} = \frac{f(x+h+t_n)-f(x)}{h+t_n}\,\frac{h+t_n}{h-t_n} < d\,\frac{h+t_n}{h-t_n} < d,$$

and hence

$$\frac{f(x+h+t_n)+f(x+h-t_n)-2f(x)}{h-t_n} < 2d.$$

Again, let $n \to \infty$. Then we get

$$\frac{f(x+h)-f(x)}{h} \leqslant d.$$

Since this is true for every $h \in (0, \delta)$, we have $\bar{f}^{+}(x) \leqslant d$. But d is an arbitrary number greater than d_0, so $\bar{f}^{+}(x) \leqslant d_0$ and (i) is proved.

It is clear that part (i) of Lemma 1 can be stated as follows:

If $E \subset (x, x+a) \subset I_0$ and E has full measure in $(x, x+a)$, then

$$\bar{f}^{+}(x) = \limsup_{t\to x,\ t\in E} \frac{f(t)-f(x)}{t-x}.$$

Also, an analogue involving $\bar{f}^{-}(x)$ holds.

THEOREM 1. *The sets $\{x\colon \bar{f}^{-}_{\mathrm{ap}}(x) < \bar{f}^{+}(x)\}$ and $\{x\colon \bar{f}^{+}_{\mathrm{ap}}(x) < \bar{f}^{-}(x)\}$ are of the first category.*

Proof. We shall show that $A = \{x\colon \bar{f}^{-}_{\mathrm{ap}}(x) < \bar{f}^{+}(x)\}$ is a set of the first category. Let Q denote the set of rationals. We set, for $r \in Q$,

$$A_r = \{x\colon \bar{f}^{-}_{\mathrm{ap}}(x) < r < \bar{f}^{+}(x)\}$$

and $\hat{A}_{rn}$ $(n = 1, 2, \ldots)$ to be the set of points x such that

$$\left|\left\{y\colon \frac{f(y)-f(x)}{y-x} > r,\ 0 < x-y < h\right\}\right| \leqslant \frac{h}{3} \quad \text{for } h \in \left(0, \frac{1}{n}\right).$$

Then $\{x\colon \bar{f}^{-}_{\mathrm{ap}}(x) < r\} \subset \bigcup\{\hat{A}_{rn}\colon n = 1, 2, \ldots\}$. Let $A_{rn} = \hat{A}_{rn} \cap A_r$. We have $A_r = \bigcup\{A_{rn}\colon n = 1, 2, \ldots\}$ and

$$\begin{aligned} A &= \bigcup\{A_{rn}\colon r \in Q,\ n = 1, 2, \ldots\} \\ &= [A \cap (I_0 \sim G)] \cup \bigcup\{A_{rn} \cap G\colon r \in Q,\ n = 1, 2, \ldots\}. \end{aligned}$$

Since $I_0 \sim G$ is known to be of the first category, it suffices to show that each $A_{rn} \cap G$ is nowhere dense. First we shall show that $\hat{A}_{rn} \cap G$ is closed relative to G so that

$$\overline{A_{rn} \cap G} \cap G \subset \hat{A}_{rn} \cap G \subset \hat{A}_{rn}.$$

Then we prove that $\overline{A_{rn} \cap G}$ contains no interval.

Let $x_0 \in G \sim \hat{A}_{rn}$ be given. Then there exists $h_0 \in (0, 1/n)$ such that

$$\left|\left\{y\colon \frac{f(y)-f(x_0)}{y-x_0} > r,\ 0 < x_0-y < h_0\right\}\right| > \frac{h_0}{3}.$$

Let $S = \{y\colon [f(y)-f(x_0)]/(y-x_0) > r,\ 0 < x_0-y < h_0\}$. Since $x_0 \in G$, $[f(y)-f(x)]/(y-x)$ for each fixed $y \in S$ is continuous at $x = x_0$ and there is an integer $k = k(y)$ such that

$$\text{(1)} \qquad \frac{f(y)-f(x)}{y-x} > r \text{ and } 0 < x-y < h_0 \text{ whenever } |x-x_0| < \frac{1}{k}.$$

For each positive integer k, let S_k be the set of $y \in S$ such that (1) holds. Clearly, $S_1 \subset S_2 \subset \ldots$ and $S = \bigcup\{S_k\colon k = 1, 2, \ldots\}$. It follows that

$$\lim_{k\to\infty}|S_k| = |S| > h_0/3,$$

and hence there exists k_0 such that $|S_{k_0}| > h_0/3$. Now, let x be any fixed point in $(x_0 - 1/k_0,\ x_0 + 1/k_0)$. We see that

$$S_{k_0} \subset \left\{y\colon \frac{f(y)-f(x)}{y-x} > r,\ 0 < x-y < h_0\right\}.$$

Therefore

$$\left|\left\{y\colon \frac{f(y)-f(x)}{y-x} > r,\ 0 < x-y < h_0\right\}\right| > \frac{h_0}{3}.$$

That is, $x \notin \hat{A}_{rn}$. We have just shown that

$$\left(x_0 - \frac{1}{k_0},\ x_0 + \frac{1}{k_0}\right) \cap G \subset G \sim \hat{A}_{rn}.$$

It follows that $G \sim \hat{A}_{rn}$ is open relative to G or, equivalently, $\hat{A}_{rn} \cap G$ is closed relative to G.

Suppose that, for some r and n, $A_{rn} \cap G$ is not nowhere dense. There exists an interval $(\alpha, \beta) \subset I_0$ such that $(\alpha, \beta) \subset \overline{A_{rn} \cap G}$. Thus

$$(\alpha, \beta) \cap G \subset \overline{A_{rn} \cap G} \cap G \subset \hat{A}_{rn}.$$

Without loss of generality, we assume that $\beta - \alpha < 1/n$. Now we fix an arbitrary point $x_0 \in (\alpha, \beta) \cap G$. For each $x \in (x_0, \beta) \cap G$, we have $x \in \hat{A}_{rn}$ and

$$\left|\left\{y\colon \frac{f(y)-f(x)}{y-x} > r,\ 0 < x-y < h\right\}\right| \leqslant \frac{h}{3} \quad \text{for } h \in \left(0, \frac{1}{n}\right).$$

In particular, for $h = x - x_0$,

$$\left|\left\{y\colon \frac{f(y)-f(x)}{y-x} > r,\ x_0 < y < x\right\}\right| \leqslant \frac{x-x_0}{3}.$$

This implies that

$$\left|G\cap\left\{y:\ \frac{f(y)-f(x)}{y-x}\leqslant r,\ x_0<y<\frac{x_0+x}{2}\right\}\right|$$
$$=\left|\left\{y:\ \frac{f(y)-f(x)}{y-x}\leqslant r,\ x_0<y<\frac{x_0+x}{2}\right\}\right|>0.$$

Hence there exists $x_1\in G$ such that

$$x_0<x_1<\frac{x_0+x}{2}\quad\text{and}\quad\frac{f(x_1)-f(x)}{x_1-x}\leqslant r.$$

Replacing x by x_1 in the above argument, we see that there exists $x_2\in G$ such that

$$x_0<x_2<\frac{x_0+x_1}{2}\quad\text{and}\quad\frac{f(x_2)-f(x_1)}{x_2-x_1}\leqslant r.$$

Moreover, we have

$$0<x_2-x_0<\frac{x_1-x_0}{2}<\frac{x-x_0}{2^2}$$

and

$$\frac{f(x_2)-f(x)}{x_2-x}=\frac{f(x_2)-f(x_1)}{x_2-x_1}\,\frac{x_2-x_1}{x_2-x}+\frac{f(x_1)-f(x)}{x_1-x}\,\frac{x_1-x}{x_2-x}\leqslant r.$$

Repeating the process, we get a sequence $\{x_m\}$ in G such that

$$0<x_m-x_0<\frac{x-x_0}{2^m}\quad\text{and}\quad\frac{f(x_m)-f(x)}{x_m-x}\leqslant r$$

for every m. Let $m\to\infty$. Since $x_0\in G$, we obtain

$$\frac{f(x_0)-f(x)}{x_0-x}\leqslant r.$$

This inequality holds for every $x\in(x_0,\beta)\cap G$. By the remark following Lemma 1, $\bar{f}^+(x_0)\leqslant r$. Thus $x_0\notin A_r$ and $x_0\notin A_{rn}\cap G$. Consequently,

$$[(\alpha,\beta)\cap G]\cap[A_{rn}\cap G]=\emptyset,$$

and hence $(\alpha,\beta)\cap G\cap\overline{A_{rn}\cap G}=\emptyset$. This contradicts the assumption that $(\alpha,\beta)\cap G\subset\overline{A_{rn}\cap G}\cap G$. The proof is completed.

Since $\bar{f}^-_{\rm ap}(x)\leqslant\bar{f}^-(x)$ and $\bar{f}^+_{\rm ap}(x)\leqslant\bar{f}^+(x)$ for all x, it is immediate from Theorem 1 that $\{x:\ \bar{f}^-(x)\neq\bar{f}^+(x)\}$ and $\{x:\ \bar{f}^-_{\rm ap}(x)\neq\bar{f}^+_{\rm ap}(x)\}$ are sets of the first category. Thus we have given another proof of Neugebauer's Theorem 10 in [3] and extended the result in [4].

LEMMA 2. *For each* $x \in I_0^0$, $H(x) \subset (0, \min\{x, 1-x\})$ *and* $H(x)$ *has full measure in this interval.*

Proof. By our definition, $H(x) = \{h > 0: x \pm h \in G\}$ and $G \subset (0, 1)$. If $h \in H(x)$, then $h > 0$ and $0 < x-h < x+h < 1$. Clearly, $H(x) \subset (0, \min\{x, 1-x\})$. Using the facts that the Lebesgue measure is invariant under translation and reflection and that G has full measure in I_0, we can show that $H(x)$ has full measure in $(0, \min\{x, 1-x\})$.

THEOREM 2. *The set* $\{x: \bar{f}^s(x) \neq \bar{f}^+(x)\}$ *is of the first category.*

Proof. Step I. The set $A = \{x: \bar{f}^s(x) < \bar{f}^+(x)\}$ is of the first category.

As in the proof of Theorem 1, we set $A_r = \{x: \bar{f}^s(x) < r < \bar{f}^+(x)\}$ for $r \in Q$ and, for each n, let $\hat{A}_{rn}$ be the set of points x such that

$$\frac{f(x+h)-f(x-h)}{2h} \leqslant r \quad \text{for } h \in \left(0, \frac{1}{n}\right) \cap H(x).$$

Also, set $A_{rn} = \hat{A}_{rn} \cap A_r$. By Lemma 1, $x \in \hat{A}_{rn}$ for some n if $\bar{f}^s(x) < r$. Hence $A_r = \bigcup\{A_{rn}: n = 1, 2, \ldots\}$ and

$$A = [A \cap (I_0 \sim G)] \cup \bigcup\{A_{rn} \cap G: r \in Q, n = 1, 2, \ldots\}.$$

If x_0 is an arbitrarily fixed point in $G \sim \hat{A}_{rn}$, then there is an $h_0 \in (0, 1/n) \cap H(x_0)$ such that

$$\frac{f(x_0+h_0)-f(x_0-h_0)}{2h_0} > r.$$

Since $h_0 \in H(x_0)$, $[f(x_0+h)-f(x_0-h)]/2h$, as a function of h, is continuous at h_0. Also,

$$h_0 \in (0, 1/n) \cap H(x_0) \subset (0, 1/n) \cap (0, \min\{x_0, 1-x_0\}).$$

There exists $\eta > 0$ such that

$$h \in \left(0, \frac{1}{n}\right) \cap (0, \min\{x_0, 1-x_0\})$$

and

$$\frac{f(x_0+h)-f(x_0-h)}{2h} > r$$

for all h satisfying $|h-h_0| < \eta$. Let

$$H = \{h \in H(x_0): |h-h_0| < \eta\}.$$

Then, for fixed $h \in H$, $[f(x+h)-f(x-h)]/2h$, as a function of x, is continuous at x_0 and there exists a positive integer $k = k(h)$ such that

$$\frac{f(x+h)-f(x-h)}{2h} > r \quad \text{for } x \in \left(x_0 - \frac{1}{k},\ x_0 + \frac{1}{k}\right). \tag{2}$$

Let H_k denote the set of $h \in H$ such that (2) holds. We see easily that $H_1 \subset H_2 \subset \ldots$ and $H = \bigcup\{H_k: k = 1, 2, \ldots\}$. Since $\{h: |h-h_0| < \eta\}$ is contained in $(0, \min\{x_0, 1-x_0\})$ in which $H(x_0)$ has full measure and $H = \{h \in H(x_0): |h-h_0| < \eta\}$, we have $|H| = |\{h: |h-h_0| < \eta\}| = 2\eta$. Therefore, there is a k_0 such that $|H_{k_0}| > \eta$ and $1/k_0 < \eta$. We want to show that the neighborhood $(x_0-1/k_0,\ x_0+1/k_0)\cap G$ of x_0 relative to G is contained in $G \sim \hat{A}_{rn}$, and thus we conclude that $\hat{A}_{rn} \cap G$ is closed relative to G. Let $x \in (x_0-1/k_0,\ x_0+1/k_0)\cap G$ be given. In view of the facts that

$$H_{k_0} \subset H \subset H(x_0) \subset (0, \min\{x_0, 1-x_0\}),$$

$H(x)$ has full measure in the interval $(0, \min\{x, 1-x\})$ (Lemma 2), and $|x-x_0| < 1/k_0 < \eta$, we obtain

$$\begin{aligned} |H_{k_0} \sim H(x)| &\leqslant |(0, \min\{x_0, 1-x_0\}) \sim (0, \min\{x, 1-x\})| \\ &< \frac{1}{k_0} < \eta < |H_{k_0}|. \end{aligned}$$

This implies that $H_{k_0} \cap H(x) \neq \emptyset$. For any $h \in H_{k_0} \cap H(x)$, we have $h \in (0, 1/n) \cap H(x)$ and

$$\frac{f(x+h)-f(x-h)}{2h} > r.$$

It follows that $x \notin \hat{A}_{rn}$, and hence $(x_0-1/k_0,\ x_0+1/k_0)\cap G \subset G \sim \hat{A}_{rn}$. Now we have, as in the proof of Theorem 1,

$$\overline{A_{rn} \cap G} \cap G \subset \hat{A}_{rn} \cap G \subset \hat{A}_{rn}.$$

Suppose that, for some r and n, there exists an interval $(\alpha, \beta) \subset I_0$ such that $\beta - \alpha < 1/n$ and $(\alpha, \beta) \subset \overline{A_{rn} \cap G}$. Then $(\alpha, \beta) \cap G \subset \hat{A}_{rn}$. Let $x_0 \in (\alpha, \beta) \cap G$ be given. As we did for Theorem 1, Step I will be proved if we show that $\bar{f}^+(x_0) \leqslant r$.

Let $S = \{x \in (x_0, \beta): \frac{1}{2}(x_0+x) \in G\}$. Then

$$\begin{aligned} |(x_0, \beta) \sim S| &= |(x_0, \beta) \sim (2G - x_0)| \\ &= 2\left|\left(\frac{x_0}{2}, \frac{\beta}{2}\right) \sim \left(G - \frac{x_0}{2}\right)\right| = 2\left|\left(x_0, \frac{x_0+\beta}{2}\right) \sim G\right| = 0 \end{aligned}$$

and

$$|(x_0, \beta) \sim (S \cap G)| \leqslant |(x_0, \beta) \sim S| + |(x_0, \beta) \sim G| = 0.$$

Hence $S \cap G$ has full measure in (x_0, β). For any $x \in S \cap G$, if we set $z = \frac{1}{2}(x_0+x)$ and $h = \frac{1}{2}(x-x_0)$, we see that

$$z \in (a, \beta) \cap G \subset \hat{A}_{rn} \quad \text{and} \quad h \in H(z)$$

since $z+h = x \in G$ and $z-h = x_0 \in G$. Therefore we have

$$\frac{f(z+h)-f(z-h)}{2h} \leqslant r, \quad \text{i.e.,} \quad \frac{f(x)-f(x_0)}{x-x_0} \leqslant r.$$

By the remark following Lemma 1, $\bar{f}^+(x_0) \leqslant r$.

Step II. The set $B = \{x: \bar{f}^+(x) < \bar{f}^s(x)\}$ is of the first category.

Just as before, we set $B_r = \{x: \bar{f}^+(x) < r < \bar{f}^s(x)\}$ for $r \in Q$. Let $\hat{B}_{rn}$ be the set of x such that

$$\frac{f(x+h)-f(x)}{h} \leqslant r \quad \text{for } h \in \left(0, \frac{1}{n}\right) \cap H(x)$$

and $B_{rn} = \hat{B}_{rn} \cap B_r$ for $n = 1, 2, \ldots$ We see that

$$B = [B \cap (I_0 \sim G)] \cup \bigcup \{B_{rn} \cap G: r \in Q, n = 1, 2, \ldots\}.$$

A similar argument as we used in Step I shows that $\hat{B}_{rn} \cap G$ is closed relative to G. We want to show that each $B_{rn} \cap G$ is nowhere dense. Suppose, for some r and n, there exists $(a, \beta) \subset I_0$ such that $\beta - a < 1/n$ and $(a, \beta) \subset \overline{B_{rn} \cap G}$. Then $(a, \beta) \cap G \subset \hat{B}_{rn}$. We fix an $x_0 \in (a, \beta) \cap G$ and let

$$p = \min\left\{\tfrac{1}{3}\min\{x_0, 1-x_0\}, x_0 - a, \beta - x_0\right\}.$$

By a moment's reflection, we find that $(0, p) \cap H(x_0) \cap \frac{1}{3}H(x_0)$ has full measure in $(0, p)$. For $h \in (0, p) \cap H(x_0) \cap \frac{1}{3}H(x_0)$, let $x_1 = x_0 - h$ and $h_1 = 2h$. We see easily that

$$x_1 \in (a, \beta) \cap G \subset \hat{B}_{rn} \quad \text{and} \quad h_1 \in H(x_1) \cap (0, 1/n).$$

Hence

$$\frac{f(x_1+h_1)-f(x_1)}{h_1} \leqslant r, \quad \text{i.e.,} \quad \frac{f(x_0+h)-f(x_0-h)}{2h} \leqslant r.$$

By Lemma 1, $\bar{f}^s(x_0) \leqslant r$. This leads to a contradiction as in the proof of Theorem 1. Step II and hence this theorem is proved.

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