COLLOQUIUM MATHEMATICUM

VOL. XXVII 1973 FASC. 2

LOCALLY COMPACT SPACES WHOSE ALEXANDROFF ONE-POINT COMPACTIFICATIONS ARE PERFECT

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All spaces are presumed to satisfy the Hausdorff separation axiom.

Let X be a completely regular space and Y a compactification of X; i.e., Y is a compact space containing (a homeomorphic copy of) X as a dense subset. Then Y is said to be a *perfect compactification of* X if, for any open subset U of X,

$$\mathrm{Fr}_Y[Y-\overline{X-U}^Y] = \overline{\mathrm{Fr}_X U}^Y,$$

where the symbols $\mathrm{Fr}_Z A$ and $\bar{A}^Z$ denote, respectively, the frontier and closure of A in Z. Skljarenko was the first to define this notion (see [2]) and proved, among other things, that a compactification Y of a completely regular space X is perfect if and only if the set $Y-X$ does not split Y at any of its points. A set N *splits the space* Y *at the point* x *of* N if x has a neighborhood U in Y such that $U\cap(Y-N) = V\cup W$, where V and W are disjoint non-void open subsets of $Y-N$ with $x\in \bar{V}^Y\cap \bar{W}^Y$.

Our main purpose is to characterize internally those locally compact, non-compact spaces whose Alexandroff one-point compactifications are perfect.

In their joint paper [1], Aarts and Van Emde Boas characterized internally those locally compact, non-compact spaces X all of whose compactifications Y have compact and connected remainders $Y-X$ in terms of a notion of a compact subset splitting a space at infinity. A compact subset C of a space X *splits the space at infinity* if $X-C = U\cup V$, where U and V are disjoint non-void open subsets of X such that $C\cup U$ and $C\cup V$ are non-compact.

THEOREM. *Let X be locally compact, non-compact space and $Y = X\cup\{\infty\}$ its Alexandroff one-point compactification. The following statements are equivalent:*

(a) *The Alexandroff one-point compactification Y of X is perfect.*

(b) *The set* $Y-X = \{\infty\}$ *does not split* Y *at the point* ∞.

(c) *No compact subset of* X *splits* X *at infinity.*

(d) *For any compactification* Z *of* X, *the remainder* $Z-X$ *is compact and connected.*

Proof. That (a) and (b) are equivalent follows from Theorem 1 of [2].

To show that (b) implies (c), suppose that condition (b) holds and suppose that there exists a compact subset C of X which splits X at infinity. Then $X-C = U \cup V$, where U and V are disjoint non-void open subsets of X with both $C \cup U$ and $C \cup V$ non-compact. By condition (b), the point ∞ cannot be in $\overline{U}^Y \cap \overline{V}^Y$ so that there is a neighborhood of ∞ in Y which does not meet one of the U and V, say U. Thus there is a compact subset K of X with $(Y-K) \cap U = \emptyset$. Let $N = X-K$ and, without loss of generality, we may assume that $K \supset C$. Hence $N \subset X-U = C \cup V$. Since $K \supset C$ and, therefore, $N \cap C = \emptyset$, $N \subset V$. Since $C \cup U$ is closed in X, we have

$$\overline{C \cup U}^X = C \cup U = X - V \subset X - N = K,$$

so that $C \cup U$ is closed in the compact set K. Hence $C \cup U$ is compact, contradicting the assumption. Thus no compact subset of X splits X at infinity.

To show that (c) implies (b), assume that condition (c) holds and suppose that the set $\{\infty\} = Y-X$ splits Y at the point ∞. Then there exists a neighborhood W of ∞ in Y such that $X \cap W = U \cup V$, where U and V are disjoint non-void open subsets of X with $\infty \in \overline{U}^Y \cap \overline{V}^Y$. Note that the set $C = X-W$ is a compact subset of X.

Suppose that one of the sets $C \cup U$ and $C \cup V$, say $C \cup U$, is compact. Then $V \cup \{\infty\}$ is a neighborhood of ∞ in Y which misses the set $C \cup U$ so that ∞ cannot be in $\overline{V}^Y \cap \overline{C \cup U}^Y$. Since $\overline{U}^Y \cap \overline{V}^Y \subset \overline{V}^Y \cap \overline{C \cup U}^Y$, the point ∞ cannot be in $\overline{U}^Y \cap \overline{V}^Y$. This, however, is a contradiction and, therefore, we must have both $C \cup U$ and $C \cup V$ non-compact. But this means that the compact set C splits X at infinity, contradicting condition (c). Hence, condition (b) holds.

The equivalence of (c) and (d) is essentially the theorem of Section 3 of [1].

The proof is now complete.

As a final remark, it is natural to ask whether the conditions of the theorem are enough to guarantee that all compactifications of such X are perfect. The following example answers this question in the negative:

Let Z be the standard unit sphere in the Euclidean 3-space, i.e.,

$$Z = \{(x, y, z) \in E^3 \mid x^2 + y^2 + z^2 = 1\},$$

and let

$$X = Z - \{(x, y, z) \in Z \mid x \geqslant 0, y \geqslant 0, z = 0\}.$$

Then Z is a compactification of X. Since X is homeomorphic to E^2, X is locally compact, non-compact and satisfies condition (c) of the theorem. However, the set $Z - X$ clearly splits the sphere Z at every point of $Z - \big(X \cup \{(1, 0, 0), (0, 1, 0)\}\big)$ so that Z is not perfect by Theorem 1 of [2].

REFERENCES

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Reçu par la Rédaction le 26. 10. 1971