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ON GENERALIZED BEURLING'S THEOREM AND SYMMETRY OF L_1-GROUP ALGEBRAS

BY

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In 1945 Beurling [1] proved the following theorem:

If f is a bounded uniformly continuous function on the real line R, then the closure in the topology of uniform convergence on compact sets of the linear span of the translates of f contains the function $\varphi(t) = \exp(i\lambda t)$.

If in Beurling's theorem an L_∞-function is taken for f and the uniform convergence on compact sets is replaced by the *-weak convergence in $L_\infty = L_1'$, the theorem thus modified is Wiener's Tauberian theorem.

We say that a function f on an arbitrary (not necessarily Abelian) locally compact group G has Wiener's Tauberian property (*W-T-property*) if the *-weak closure of the linear span of the right translates of f contains a normalized extreme continuous positive-definite function.

In [3] Gel'fand and Naĭmark proved what they call a *generalized Beurling's theorem*. If A is a Banach *-algebra with the unit element, then A is symmetric (i.e., $\mathrm{Sp}\, x^*x \geqslant 0$ for x in A) if and only if for every linear functional F on A the *-weak closure of $\{F_a\colon a \in A\}$, where $\langle x, F_a\rangle = \langle ax, F\rangle$, contains an extreme positive normalized functional.

Suppose that $L_1(G)$ is symmetric. Then so is the algebra $A = Ce + L_1(G)$, where e is the unit element of A. Consequently, by the Gel'fand-Naĭmark theorem, for every L_∞-function f on G the *-weak closure of the set $\{f * a\colon a \in L_1(G)\}$ contains a continuous positive-definite function φ. However, we do not know any simple way of showing that φ may be non-zero, even in the case if G is Abelian. In fact, in the latter case such a φ does exist but the proof requires additional properties of the algebra $L_1(G)$ (e.g., regularity). Thus the theorem of Gel'fand and Naĭmark does not readily yield neither the quoted Beurling's nor Wiener's Tauberian theorem (saying that every L_∞ function on R has W-T property). The aim of this paper is to study an analogue of the "generalized Beurling theorem" for locally compact (may be non-abelian) groups, and to point out connections between a version of it and the symmetry of the L_1-group algebra.

The paper is organized as follows. After preliminary section 1 we define class $\boldsymbol{F}(G) = \boldsymbol{F}$ of functions in $L_\infty(G)$ such that for each f in $\boldsymbol{F}$ there is u in $L_1(G)$ with $u * f = f$. The aim of section 2 is to prove that the symmetry of $L_1(G)$ is equivalent to the fact that each f in $\boldsymbol{F}$ have the W-T-property. In section 3 the class $\boldsymbol{F}$ is studied more closely, its characterisation is given in the case of compact groups and a theorem is provided for a general case. Then in section 4 some attempts are made to characterize the class W-T. For instance, it is proved that almost periodic functions are in W-T. Finally, in section 5, we make some remarks on the preservation of $\boldsymbol{F}$, W-T and the symmetry of L_1 under taking homomorphic images and subgroups.

1. Preliminaries. Let G be a locally compact topological group. The differential of the left-invariant Haar measure is denoted by ds, and the Radon-Nikodym derivative of the right-invariant Haar measure with respect to the left-invariant Haar measure is denoted by $\Delta(s)$. Let $L_1(G)$ denote the Banach $*$-algebra with the norm, multiplication, and involution defined by

$$\|x\| = \int |x(s)|\,ds, \qquad x * y(t) = \int x(s)y(s^{-1}t)\,ds, \qquad x^*(t) = \Delta(t)^{-1}x(t^{-1}),$$

respectively. The dual space $L_\infty(G)$ of $L_1(G)$ consists of the essentially bounded measurable functions on G and each f in $L_\infty(G)$ defines a functional on $L_1(G)$ by

$$\langle x, f\rangle = \int_G x(s)f(s^{-1})\,ds.$$

The following equalities are easy to verify

$$\langle a * x, f\rangle = \langle x, f * \hat{a}\rangle, \qquad \text{where } \hat{a}(s) = \Delta(s)a(s), \tag{1.1}$$

$$\langle a * x, f\rangle = \langle a, x * f\rangle. \tag{1.2}$$

Let $\boldsymbol{A}(G)$ be equal to $L_1(G)$, if G is discrete, and to $Ce + L_1(G)$, if G is not discrete. Most of the results in the present paper are much easier to prove if G is discrete, so we shall focus our attention on the non-discrete case and assume throughout that

$$\boldsymbol{A}(G) = Ce + L_1(G).$$

The dual space $\boldsymbol{A}'(G)$ of $\boldsymbol{A}(G)$ is then obviously equal to $C \times L_\infty(G)$, and if $F = (a, f)$, then $\langle \lambda e + x, F\rangle = a\lambda + \langle x, f\rangle$.

For f in $L_\infty(G)$ let $\beta(f)$ denote the $*$-weak closure of the linear span of the right translates of f. It is easy to show that

$$\beta(f) = \{f * \hat{a}:\ a \in L_1(G)\}^-,$$

where the closure is taken in the $*$-weak topology in $L_\infty(G)$.

Let us recall (cf. [5], p. 191) that a left ideal I in $L_1(G)$ is called *regular* if there exists u in $L_1(G)$ such that for every x in $L_1(G)$ we have $x - x * u \in I$.

For every regular left ideal I in $L_1(G)$ there is a (proper) left ideal I' in $A(G)$ such that

$$I = I' \cap L_1(G)$$

(cf. [5], p. 192). Clearly, I' contains an element $e + x$ with x in $L_1(G)$. A regular left ideal I in $L_1(G)$ is a non-trivial subspace of $L_1(G)$ and, consequently, there is a non-zero function f in $L_\infty(G)$ such that

$$\langle x, f\rangle = 0 \quad \text{for } x \text{ in } I. \tag{1.3}$$

Let F denote the set of the non-zero functions in $L_\infty(G)$ such that for each f in F there is a regular left ideal I for which (1.3) holds. The following proposition describes the class F somewhat better.

PROPOSITION 1.1. *We have $f \in F$ if and only if there is a u in $L_1(G)$ such that*

$$f = u * f \quad \textit{almost everywhere.} \tag{1.4}$$

Proof. If I is a left regular ideal, then there is a u in $L_1(G)$ such that, for every x in $L_1(G)$, $x - x * u \in I$. Consequently, for every x in $L_1(G)$ we have

$$\langle x - x * u, f\rangle = \langle x, f\rangle - \langle x * u, f\rangle = \langle x, f\rangle - \langle x, u * f\rangle = \langle x, f - u * f\rangle = 0,$$

whence $f - u * f = 0$ a.e.

Conversely, if f satisfies (1.4), then the set

$$I = \{x \in L_1(G):\ \langle a * x, f\rangle = 0 \quad \text{for each } a \text{ in } L_1(G)\}$$

is a left ideal. Since $f = u * f$ a.e., for each x in $L_1(G)$ we have $\langle a * x - a * x * u, f\rangle = 0$, which means that $x - x * u \in I$ and, consequently, I is regular.

PROPOSITION 1.2. *If a hermitian element $\lambda e + x$ in $A(G)$ is not invertible in $A(G)$, then there exists a function f in F such that*

$$\lambda f + x * f = 0 \ \ a.e.$$

Proof. If a hermitian element $y = \lambda e + x$ is not invertible, then $A(G)y = I$ is a proper left ideal. Consequently, there is a non-zero functional $F = (a, f)$ in $A'(G)$ such that $\langle z, F\rangle = 0$ for z in I. For every element u in $L_1(G)$ we then have $\lambda u + u * x \epsilon I$, whence $\langle \lambda u + u * x, f\rangle = \langle u, \lambda f + x * f\rangle = 0$, which implies $\lambda f + x * f = 0$ a.e.

2. Now we are going to prove that $L_1(G)$ is symmetric if and only if the functions in the class $\boldsymbol{F}$ have W-T-property.

It is well known (cf. [5], p. 357) that if $\boldsymbol{A}$ is a Banach *-algebra with the unit element, then for every left ideal I there exists a normalized positive functional F such that $\langle x, F\rangle = 0$ for x in I. However, it is not known whether this is true for symmetric Banach *-algebras without unit, although this is the case if I is regular. For $L_1(G)$ we have

LEMMA 2.1. *If $L_1(G)$ is symmetric, then for every regular left ideal I in $L_1(G)$ there exists a continuous normalized positive-definite function φ such that*

$$\langle x, \varphi\rangle = 0 \quad \textit{for } x \textit{ in } I. \tag{2.1}$$

Proof. Since I is regular, there exists an ideal I' in $\boldsymbol{A}(G)$ containing I and an element $e+y$ with y in $L_1(G)$. Let Φ be a positive normalized functional on $\boldsymbol{A}(G)$ which annihilates I'. Clearly, Φ does not annihilate $L_1(G)$, since $\langle y, \Phi\rangle = -1$. Therefore there is a continuous normalized positive-definite function φ on G such that $\Phi = (1, \varphi)$. Since $I \subset I' \cap L_1(G)$, we have $\langle x, \varphi\rangle = 0$ for x in I.

Remark 2.2. The set of normalized continuous positive-definite functions which satisfy (2.1) is of course *-weakly compact and convex, and therefore, by the Kreĭn-Mil'man theorem, it contains extreme points, which turn out to be extreme in the set of all continuous normalized positive-definite functions. In fact, if $\varphi = \varphi_1 + \varphi_2$ and φ annihilates I, then $\langle x^* * x, \varphi_1\rangle = \langle x^* * x, \varphi_2\rangle = 0$ for all x in I and hence, by the Cauchy's inequality, $|\langle x, \varphi_1\rangle| = |\langle x, \varphi_2\rangle| = 0$.

Similarly, the existence of a continuous normalized positive-definite function in $\beta(f)$ implies the W-T-property for f.

THEOREM. *$L_1(G)$ is symmetric if and only if every function f in $\boldsymbol{F}$ has the W-T-property.*

Proof. Suppose $L_1(G)$ is symmetric and $f \epsilon \boldsymbol{F}$. By proposition 1.1, the left ideal

$$I_f = \{x \epsilon L_1(G)\colon \langle a * x, f\rangle = 0 \text{ for all } a \text{ in } L_1(G)\}$$

is regular. Consequently, Lemma 2.1 implies that there exists a continuous extreme normalized positive-definite function φ such that $\langle x, \varphi\rangle = 0$ for x in I_f.

To prove that $f \epsilon$ W-T, it suffices to show that the function φ belongs to the set $\beta(f)$. To see the latter, observe that if this were not the case, then, by the Hahn-Banach theorem, there would exist y in $L_1(G)$ such that $\langle y, \varphi\rangle = 1$ and $\langle y, f * \check{a}\rangle = 0$ for all a in $L_1(G)$. But the last equality means, by the definition of I_f, that y is in I_f, so $\langle y, \varphi\rangle = 0$, which is a contradiction.

Now suppose that $L_1(G)$ is not symmetric. Then there is an element x in $L_1(G)$ such that $e + x^* * x$ is not invertible in $\boldsymbol{A}(G)$. Thus, by proposition 1.2, there is an f in $\boldsymbol{F}$ such that

$$f + x^* * x * f = 0 \text{ a.e.}$$

If f has W-T-property, then there is a net $f * \hat{a}_\gamma$ in $\beta(f)$, which is *-weakly convergent to a continuous normalized positive-definite function φ. Clearly,

$$f * \hat{a}_\gamma + x^* * x * f * \hat{a}_\gamma = 0 \quad \text{a.e. for all } \gamma,$$

whence

$$\varphi + x^* * x * \varphi = 0 \text{ a.e.} \tag{2.2}$$

But, since φ is continuous, (2.2) holds everywhere, and this is impossible, because

$$\varphi(e) + x^* * x * \varphi(e) = \varphi(e) + \langle x^* * x, \varphi\rangle = 1 + \langle x^* * x, \varphi\rangle \geqslant 1.$$

3. Now we are going to prove several propositions which describe functions in $\boldsymbol{F}$.

A function f in $L_\infty(G)$ is called *finitely dimensional* if $\beta(f)$ is a finite-dimensional linear space.

Note first that if f is finitely dimensional, then f is continuous and, moreover, $\beta(f)$ is simply the linear span of finitely many translates of f. In fact, if $\{e_\gamma\}$ is a sequence of approximate units in $L_1(G)$, then, for each γ, $f * \hat{e}_\gamma$ is continuous and belongs to $\beta(f)$. Clearly, $\{f * \hat{e}_\gamma\}$ is *-weakly convergent to f but since all topologies in a finitely dimensional linear space are equivalent, $\{f * \hat{e}_\gamma\}$ converges to f uniformly, and so f is continuous.

PROPOSITION 3.1. *Every finitely dimensional function f (in $L_\infty(G)$) belongs to $\boldsymbol{F}$.*

Proof. Let f be a finitely dimensional function and let $f_1, \ldots, f_n$ be a basis for $\beta(f)$. Then

$$f^s(t) = f(ts) = \sum_{i=1}^{n} a_i(s) f_i(t).$$

Note that

$$f(s) = f^s(e) = \sum_{i=1}^{n} a_i(s) f_i(e), \tag{3.1}$$

$$\begin{aligned} a * f(s) &= \int_G a(t) f(t^{-1}s)\,dt \\ &= \sum_{i=1}^{n} a_i(s) \int_G a(t) f_i(t^{-1})\,dt = \sum_{i=1}^{n} a_i(s)\,\langle a, f_i\rangle. \end{aligned} \tag{3.2}$$

Since f_i, $i = 1, \ldots, n$, are linearly independent, there exist $a_1', \ldots, a_n'$ in $L_\infty'(G)$ such that

$$\langle a_j', f_i \rangle = \sigma_{ij} \quad 1 \leqslant i, j \leqslant n.$$

But since $\beta(f)$ is finite dimensional, the functionals a_i' can be chosen in $L_1(G)$. Let then $a_1, \ldots, a_n$ belong to $L_1(G)$ and

$$\langle a_j, f_i \rangle = \delta_{ij}, \quad 1 \leqslant i, j \leqslant n.$$

Putting

$$u = \sum_{i=1}^{n} f_i(e) a_i,$$

note that $\langle u, f_i \rangle = f_i(e)$, whence, by (3.2),

$$u * f(s) = \sum_{i=1}^{n} a_i(s) f_i(e) = f(s).$$

PROPOSITION 3.2. *Let G be a compact group. Then $f \in \boldsymbol{F}$ if and only if f is finitely dimensional.*

Proof. If G is compact, then for a in $L_1(G)$ the operator

$$T_a\colon L_\infty(G) \ni f \to a * f \in L_\infty(G)$$

is compact. If $f \in \boldsymbol{F}$, then $u * f = f$ for each u in $L_1(G)$, which means that f is an eigenvector with eigenvalue 1 of the operator T_u. Clearly, for each s in G, $(T_a f)^s = T_a f^s$, whence f^s is also an eigenvector with eigenvalue 1 of T_u. Hence, the space spanned by right translates of f is contained in the eigenspace corresponding to the eigenvalue 1 of T_u, which is finitely dimensional, since T_u is compact.

4. Now turn to the functions f which have the Wiener's Tauberian property (symbolically, $f \in$ W-T).

In virtue of Theorem 1, there are functions f (even in $\boldsymbol{F}$) which do not have the W-T-property if only $L_1(G)$ is not symmetric. We suspect that the symmetry of $L_1(G)$ implies that every $L_\infty(G)$-function has the W-T-property ($L_\infty(G) \in$ W-T) but we are unable to prove it (**P 746**).

The following proposition with an easy proof is very useful in applications:

PROPOSITION 4.1. *If $f \in L_\infty(G)$ and $a \in L_1(G)$, then $f * \hat{a} \in W$-T implies $f \in W$-T.*

Proof. If a net $\{f * \hat{a} * \hat{a}_\gamma\}$ is $*$-weakly convergent to a continuous positive-definite extreme function φ and $b_\gamma = a * a_\gamma$, then the net $\{f * \hat{b}_\gamma\}$ is convergent to φ, since $\widehat{a * a_\gamma} = \hat{a} * \hat{a}_\gamma$.

PROPOSITION 4.2. *If $f \in L_1(G) \cap L_2(G)$, then $f \in W\text{-}T$.*

Proof. Since $f * \hat{f^*} = f * f^\sim$ and $f * f^\sim$ is positive-definite (cf. [4]) and continuous, the result follows by remark 2.2.

PROPOSITION 4.3. *If G is compact, then $L_\infty(G) \in W\text{-}T$.*

Proof. By proposition 4.2.

In virtue of proposition 4.3 and Theorem 1 we have

COROLLARY 4.4. (van Dijk). *If G is compact, then $L_1(G)$ is symmetric.*

LEMMA 4.5. *If G is a compact group, $\tilde{G}$ a dense subset of G and f, g are continuous functions on G, then $f * g$ is the limit of a uniformly convergent sequence of linear combinations of functions f^t, $t \in \tilde{G}$.*

Proof. Without loss of generality we may assume that g is real non-negative and $\int g(t)\,dt = 1$. Since G is compact, $\{f^t\colon t \in \tilde{G}\}$ is a conditionally compact set in $C(G)$ and the set

$$C = \operatorname{conv}\{f^t\colon t \in \tilde{G}\}^- = \operatorname{conv}\{f^t\colon t \in G\}^-$$

is (strongly) compact in $C(G)$. Consider the function

$$F\colon G \ni t \to f^t \in C.$$

Clearly, F is strongly continuous, and since C is compact, it is Bochner integrable with respect to any probability measure on G (cf. [6]). But

$$f * g = \int F(t) g(t^{-1})\,dt \in C,$$

the last inclusion being an immediate consequence of the definition of the Bochner integral.

PROPOSITION 4.6. *If f is an almost periodic function on a locally compact group G, then $f \in W\text{-}T$.*

Proof. By the definition, f is an almost periodic function if the set

$$A = \{f^t\colon t \in G\}$$

is relatively compact (in the norm topology) in $L_\infty(G)$.

Let $\tilde{G}$ be the group of operators on the compact set $\bar{A}$. Consider

$$T_s\colon \bar{A} \ni g \to g^s \in \bar{A}.$$

Since $\|g^s\|_\infty = \|g\|_\infty$, T_s is an isometry and, consequently, $\tilde{G}$ is a relatively compact group in the metric

$$\varrho(T_s, T_t) = \max_{h \in \bar{A}} \|T_s h - T_t h\|_\infty.$$

Let $\bar{G}$ be the completion of $\tilde{G}$ in ϱ and a be the homomorphism

$$a\colon G \ni s \to T_s \in \bar{G}.$$

Let $H = \ker a$. The function f is constant on the cosets $sH = Hs$.

Since

$$|f(t)-f(s)| \leqslant \|f^t - f^s\|_\infty \leqslant \varrho(T_t, T_s),$$

the function defined by $\dot{f}(T_t) = f(t)$ is uniformly continuous on a dense subset $\tilde{G}$ of $\bar{G}$ and, consequently, it extends to a continuous function $\dot{f}$ on $\bar{G}$.

By Lemma 4.5, the positive-definite function $\dot{f} * \dot{f}^\sim$ on $\bar{G}$ is the limit of a uniformly convergent sequence of linear combinations of right-translations of $\dot{f}$ by the elements of $\tilde{G}$. This remains true for the positive-definite function $\varphi = \dot{f} * \dot{f}^\sim | \tilde{G}$ and thus $\varphi \circ a$ is a positive-definite function on G belonging to the uniform closure of linear combinations of right translations of f.

5. In this section we prove a few facts on preservation of the properties W-T, F and the symmetry of $L_1(G)$ under taking homomorphic images and subgroups. We start with the following remarks.

Let G be a locally compact group, H a normal subgroup of G and let

$$a\colon G \to G/H$$

be the natural homomorphism. Let, further, the measures ds, dh, $d\dot{s}$ on G, H and G/H be adjust in such a way that

$$\int_G x(s)\,ds = \int_{G/H}\int_H x(sh)\,dh\,d\dot{s}.$$

The homomorphism a determines a homomorphism

$$\eta\colon L_1(G) \ni x \xrightarrow{\text{onto}} \dot{x} \in L_1(G/H)$$

defined by

$$\dot{x}(sH) = \int_H x(sh)\,dh.$$

To see that η is "onto", one can use a well-known fact (cf. [2]) that there is a continuous non-negative function on G such that

$$\int_H y(sh)\,dh = 1.$$

Then for each $x \in L_1(G/H)$ the function $z(s) = x(a(s))y(s)$ is in $L_1(G)$ and $\eta(z) = x$. The adjoint mapping $\eta^*\colon L_\infty(G/H) \to L_\infty(G)$ is then, clearly, given by

$$\eta^* f(s) = f \circ a(s), \qquad s \in G.$$

LEMMA 5.1. *For any f in $L_\infty(G/H)$ we have*

$$\beta(\eta^* f) = \eta^* \beta(f).$$

Proof. Clearly,

$$(\eta^*f)_t(s) = \eta^*f(st) = f\circ a(st) = f\big(a(s)a(t)\big) = \eta^*f_{a(t)}(s),$$

where $t, s\in G$, which means that η^* intertwines right translations. Since η is onto, η^* commutes also with the $*$-weak closure (in $L_\infty(G/H)$ and $L_\infty(G)$, respectively), and this completes the proof.

PROPOSITION 5.2. *If $f\in L_\infty(G/H)$ and η^*f has property W-T, then f has property W-T.*

Proof. Let $f\in L_\infty(G/H)$. Then $\eta^*f\in L_\infty(G)$ and, by assumption, η^*f has property W-T. Hence, by Lemma 5.1, there is a positive-definite continuous normalized function φ in $\beta(\eta^*f) = \eta^*\beta(f)$. Let $\varphi = \eta^*\dot{\varphi} = \dot{\varphi}\circ a$ with $\dot{\varphi}\in\beta(f)$. Clearly, $\dot{\varphi}$ is a continuous positive-definite normalized function on G/H, and this completes the proof.

LEMMA 5.3. *We have*

$$\eta^*\colon \boldsymbol{F}(G/H) \xrightarrow{\text{into}} \boldsymbol{F}(G).$$

Proof. In fact, let $f\in\boldsymbol{F}(G/H)$. By the definition, $u*f = f$ for some u in $L_1(G/H)$. Let $x\in L_1(G)$ and $\eta x = u$. We claim that $x*\eta^*f = \eta^*f$, which means that $\eta^*f\in\boldsymbol{F}(G)$. In fact, for y in $L_1(G)$ we have

$$\langle y, x*\eta^*f\rangle = \langle y*x, \eta^*f\rangle = \langle \eta y*u, f\rangle = \langle \eta y, f\rangle = \langle y, \eta^*f\rangle.$$

PROPOSITION 5.4. *If $L_1(G)$ is symmetric, then so is $L_1(G/H)$.*

Proof. We prove that every function f in $\boldsymbol{F}(G/H)$ has property W-T. In fact, by Lemma 5.3, $\eta^*f\in\boldsymbol{F}(G)$ and, by Theorem 1, it has property W-T, since $L_1(G)$ is symmetric. Consequently, by proposition 5.2, f has property W-T.

PROPOSITION 5.5. *Let H be a closed subgroup of a locally compact group G. If every function f in $L_\infty(G)$ has property W-T, then $L_1(H)$ is symmetric.*

Proof. We begin with three easy facts the proofs of which are routine:

(i) If u is a continuous function with compact support on G and if $\varphi\in L_\infty(H)$ and

$$\psi(t) = \varphi*u(t) \overset{\text{df}}{=} \int_H \varphi(h)u(h^{-1}t)\,dh,$$

then $\psi\in L_\infty(G)$ is a continuous function.

(ii) If $z\in L_1(H)$, φ is a bounded continuous function on H and u is a continuous function with compact support on G, then

$$(z*\varphi)*u(t) = z*(\varphi*u)(t).$$

(iii) If $z \in L_1(H)$, ψ is a bounded continuous function on G and $a \in L_1(G)$, then

$$(z * \psi) * \hat{a}(t) = z * (\psi * \hat{a})(t).$$

Now suppose to the contrary that $L_1(H)$ and, consequently, $A(H)$ is not symmetric. Let $e + x^* * x$, $x \in L_1(H)$, be an element not invertible in $A(H)$. By proposition 1.2, there is a function φ in $F(H)$ such that

$$\varphi + x^* * x * \varphi = 0.$$

Let u be a continuous function with compact support on G and let $\psi = \varphi * u$. Clearly, $\psi \in L_\infty(G)$ and

$$\psi + x^* * x * \psi = (\varphi + x^* * x * \varphi) * u = 0.$$

By assumption, ψ has property W-T and, consequently, there is a net $\{a_\gamma\}$, where $a_\gamma \in L_1(G)$, such that $\psi * \hat{a}_\gamma$ is *-weakly convergent to a continuous normalized positive-definite function φ_0. By (iii), for every γ we have

$$0 = (\psi + x^* * x * \psi) * \hat{a}_\gamma = \psi * \hat{a}_\gamma + x^* * x * (\psi * \hat{a}_\gamma),$$

whence

$$\varphi_0 + x^* * x * \varphi_0 = 0.$$

But the restriction of φ_0 to H is a positive-definite normalized continuous functions on H, whence

$$\varphi_0(e) + x^* * x * \varphi_0(e) = \varphi_0(e) + \langle x^* * x, \varphi_0 \rangle \geqslant 1,$$

which is a contradiction.

6. The following problems seem to remain open. Let H be a closed subgroup of G.

1. If $L_1(G)$ is symmetric, is then $L_1(H)$ symmetric? (**P 747**)

2. If f is a continuous bounded function on G and $f \in$ W-T, is it true that $f|H \in$ W-T (relatively to H)? (**P 748**)

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