COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1976 FASC. 2

GENERALIZED P_0-LATTICES OF ORDER ω^+

BY

T. TRACZYK AND W. ZARĘBSKI (WARSZAWA)

In 1964, the first author introduced in [6] the notion of P_0-lattices. Epstein and Horn [1] investigated the theory of P_0-lattices in detail and they used this concept in searching for some new important generalizations of the Post algebra of finite order. In this way they discovered P_1-lattices and P_2-lattices.

The aim of the present paper is to show that hypothesis of fixed finite order in the theory of P_0-lattices can be widely replaced by much weaker one. In a separate paper we will study generalized P_1-lattices and P_2-lattices.

1. Preliminaries. Let A be a bounded distributive lattice. The least and the greatest elements of A will be denoted by 0 and 1, respectively; $x\cup y$ and $x\cap y$ (sometimes xy) will stand for the join and the meet of x and y, respectively, in A. By the *center* of A we understand the sublattice B of all complemented elements of A. The complement of $a\in A$, if it exists, is denoted by $\bar{a}$. The greatest element $z\in A$ such that $xz\leqslant y$, if it exists, is denoted by $x\to y$ and it is called a *relative pseudo-complement* of x with respect to y. A is said to be a *Heyting algebra* if $x\to y$ exists for arbitrary $x, y\in A$. A Heyting algebra A is said to be an *L-algebra* if

$$(x\to y)\cup(y\to x) = 1 \quad \text{for all } x, y\in A.$$

A bounded distributive lattice A is called *pseudo-complemented* if $\neg x = x\to 0$ exists for every $x\in A$. A is a *Stone lattice* if the identity $\neg x\cup\neg\neg x = 1$ holds for each $x\in A$ [2].

The greatest Boolean element z of a bounded distributive lattice A such that $xz\leqslant y$ will be denoted by $x\Rightarrow y$. A lattice A is called a *B-algebra* if $x\Rightarrow y$ exists for all $x, y\in A$. A B-algebra is said to be a *P-algebra* if

$$(x\Rightarrow y)\cup(y\Rightarrow x) = 1 \quad \text{for all } x, y\in A.$$

These definitions are in [1]. The identity

$$x\Rightarrow(y\cup z) = (x\Rightarrow y)\cup(x\Rightarrow z)$$

holds for all x, y, z in P-algebra. Similarly,

$$x \to (y \cup z) = (x \to y) \cup (x \to z)$$

holds in an L-algebra.

2. P_0-lattices of order ω^+. Let A be a bounded distributive lattice with the center B. Let

$$0 = e_0 \leqslant e_1 \leqslant \ldots \leqslant e_\omega = 1 \tag{2.1}$$

be a countable chain of elements of A. $\langle A; (e_i)_{0\leqslant i\leqslant \omega}\rangle$ is said to be a P_0-*lattice of order* ω^+ if, for every $x \in A$, there exists a monotonic sequence $x_1 \geqslant x_2 \geqslant \ldots$ of elements of the center B such that

$$x = \bigcup_{i=1}^{\infty} x_i e_i. \tag{2.2}$$

Then (2.2) is called a *monotonic representation* of x. A P_0-lattice $\langle A; (e_i)_{0\leqslant i\leqslant \omega}\rangle$ is called an FP_0-*lattice of order* ω^+ if, for every $x \in A$, there exists a monotonic representation (2.2) such that

$$x_n = x_{n+1} = \ldots = x_\omega \quad \text{for some } n \geqslant 1. \tag{2.3}$$

In both cases $(e_i)_{0\leqslant i\leqslant \omega}$ is called a *chain base* of A. For the sake of brevity, if no confusion is possible, we will write a P_0-lattice A of order ω^+ instead of a P_0-lattice $\langle A; (e_i)_{0\leqslant i\leqslant \omega}\rangle$ of order ω^+, and an FP_0-lattice instead of an FP_0-lattice of order ω^+. The following lemmas will be useful in the sequel.

LEMMA 2.1 (see, e.g., [4]). *If, in a lattice A, there exists the least upper bound $\bigcup_{t\in T} a_t$ and there exists $a \to c$ for every $c \in A$, then there exists the least upper bound $\bigcup_{t\in T}(a \cap a_t)$ and the following identity holds:*

$$a \cap \bigcup_{t\in T} a_t = \bigcup_{t\in T}(a \cap a_t).$$

LEMMA 2.2 (see, e.g., [1]). *If A is a bounded distributive lattice, and b belongs to the center of A, then for every $c \in A$ there exists $b \to c$ and*

$$b \to c = \bar{b} \cup c. \tag{2.4}$$

LEMMA 2.3 (see [5]). *If A is a P_0-lattice of order ω^+ and* (2.2) *is a monotonic representation of x, then*

$$x e_k = \bigcup_{i=1}^{k} x_i e_i \quad \textit{for every } k \geqslant 1. \tag{2.5}$$

LEMMA 2.4. *If A is a P_0-lattice of order ω^+, then*

$$\bigcup_{i=1}^{\infty} e_i = 1.$$

The proof is obvious.

LEMMA 2.5. *If A is an FP_0-lattice, then each $x \in A$ has a finite monotonic representation of the form*

$$x = \bigcup_{i=1}^{n} x_i e_i \cup x_\omega, \quad \text{where } n \text{ depends on } x. \tag{2.6}$$

The main result of this section is the following

THEOREM 2.1. *Let A be a P_0-lattice of order ω^+ and let $x_1 \geqslant x_2 \geqslant \ldots$ be a monotonic sequence of elements of the center B of A. Then the least upper bound on the left-hand side of formula* (2.7) *exists if and only if the greatest lower bound on the right-hand side exists. If they both exist, the equality*

$$\bigcup_{i=1}^{\infty} x_i e_i = \bigcap_{i=1}^{\infty} (x_i \cup e_{i-1}) \tag{2.7}$$

holds. In the case of an FP_0-lattice formula (2.7) *reduces to*

$$\bigcup_{i=1}^{n} x_i e_i \cup x_\omega = \bigcap_{i=1}^{n} (x_i \cup e_{i-1}). \tag{2.7'}$$

Proof. Suppose that the left-hand side of (2.7) exists and that

$$x = \bigcup_{i=1}^{\infty} x_i e_i. \tag{i}$$

For a fixed i we have

$$x_i e_i \leqslant \begin{cases} x_j & \text{if } j \leqslant i, \\ e_{j-1} & \text{if } j > i. \end{cases} \tag{ii}$$

Hence $x_i e_i \leqslant x_j \cup e_{j-1}$ for all $i, j \geqslant 1$, i.e. $x \leqslant x_j \cup e_{j-1}$ for $j = 1, 2, \ldots$ Now let $z \leqslant x_j \cup e_{j-1}$ for $j = 1, 2, \ldots$ and for a certain $z \in A$. We have to show that $z \leqslant x$. To do this let

$$z = \bigcup_{k=1}^{\infty} z_k e_k$$

be a monotonic representation. Since $z_k e_k \leqslant z \leqslant x_j \cup e_{j-1}$ for all j and k, we have

$$z_k e_k \leqslant x_k \cup \bigcup_{j=1}^{k} x_j e_j$$

by induction argument. Therefore,

$$z_k e_k = (z_k e_k) e_k \leqslant \bigcup_{i=1}^{k} x_i e_i \leqslant x \quad \text{for } k = 1, 2, \ldots$$

Consequently, $z \leqslant x$, i.e. (2.7) holds.

Let us now suppose that, for a given monotonic sequence $x_1 \geqslant x_2 \geqslant \ldots$ of elements of the center B, there exists the greatest lower bound

$$y = \bigcap_{i=1}^{\infty} (x_i \cup e_{i-1}).$$

By virtue of (ii),

$$\text{(iii)} \qquad x_i e_i \leqslant y \quad \text{for } i = 1, 2, \ldots$$

Suppose further that, for a certain $z \in A$, the inequality $x_i e_i \leqslant z$ holds for $i = 1, 2, \ldots$ It remains to show that $y \leqslant z$. Since, by induction argument,

$$\text{(iv)} \qquad e_k \bigcap_{i=1}^{k} (x_i \cup e_{i-1}) \leqslant z \quad \text{for } k = 1, 2, \ldots,$$

we have

$$e_k y \leqslant e_k \bigcap_{i=1}^{k} (x_i \cup e_{i-1}) \leqslant z \quad \text{for every } k.$$

Using a monotonic representation

$$y = \bigcup_{k=1}^{\infty} y_k e_k,$$

we conclude that

$$y_k e_k = (y_k e_k) e_k \leqslant y e_k \leqslant z \quad \text{for } k = 1, 2, \ldots,$$

i.e. $y \leqslant z$, and this completes the proof.

COROLLARY 2.1. *If*

$$x = \bigcup_{i=1}^{\infty} x_i e_i \quad \textit{and} \quad y = \bigcup_{j=1}^{\infty} y_j e_j$$

are monotonic representations of elements x and y, respectively, in a P_0-lattice A of order ω^+, then

$$x \cup y = \bigcup_{i=1}^{\infty} (x_i \cup y_i) e_i \quad \textit{and} \quad x \cap y = \bigcup_{i=1}^{\infty} (x_i \cap y_i) e_i$$

are monotonic representations of $x \cup y$ and $x \cap y$, respectively.

Proof. The former identity is obvious. In order to prove the latter, let us put x and y in the dual forms

$$x = \bigcap_{i=1}^{\infty} (x_i \cup e_{i-1}) \quad \text{and} \quad y = \bigcap_{j=1}^{\infty} (y_j \cup e_{j-1})$$

in virtue of Theorem 2.1. Then the following inequalities are obvious:

$$x \cap y \leqslant (x_i \cup e_{i-1}) \cap (y_i \cup e_{i-1}) \quad \text{for } i = 1, 2, \ldots$$

On the other hand, if, for a certain $z \in A$,

$$z \leqslant (x_i \cup e_{i-1}) \cap (y_i \cup e_{i-1}) \quad \text{for } i = 1, 2, \ldots,$$

then $z \leqslant x$ and $z \leqslant y$. So we proved that

$$x \cap y = \bigcap_{i=1}^{\infty} [(x_i \cup e_{i-1}) \cap (y_i \cup e_{i-1})].$$

Therefore

$$x \cap y = \bigcap_{i=1}^{\infty} (x_i y_i \cup e_{i-1}) = \bigcup_{i=1}^{\infty} x_i y_i e_i.$$

3. Pseudo-complements in a P_0-lattice of order ω^+. Let A be a P_0-lattice of order ω^+. In this section we study the following questions:

1. Under what conditions is A a Stone lattice?
2. Under what conditions is A an L-algebra?
3. Under what conditions is A a P-algebra?

LEMMA 3.1. *Let*

$$x = \bigcup_{i=1}^{\infty} x_i e_i \quad \text{and} \quad y = \bigcup_{j=1}^{\infty} y_j e_j$$

be monotonic representations of $x \in A$ and $y \in A$, respectively.

(i) *Assume that $e_i \to y$ exists for all $i > 0$. Then $x \to y$ exists if and only if the infinite meet*

$$\bigcap_{i=1}^{\infty} [\bar{x}_i \cup (e_i \to y)]$$

exists. If both exist, then

$$x \to y = \bigcap_{i=1}^{\infty} [\bar{x}_i \cup (e_i \to y)]. \tag{3.1}$$

(ii) *Assume that $x \to e_i$ exists for all i. Then $x \to y$ exists if and only if the infinite meet*

$$\bigcap_{j=1}^{\infty} [y_j \cup (x \to e_{j-1})]$$

exists. If both exist, then

$$x \to y = \bigcap_{j=1}^{\infty} [y_j \cup (x \to e_{j-1})]. \tag{3.2}$$

Proof. (i) Suppose

$$z = \bigcap_{i=1}^{\infty} [\bar{x}_i \cup (e_i \to y)]$$

exists in A. We show that $x \to y$ exists. Let $\bigcup_{i=1}^{\infty} u_i e_i$ be a monotonic representation of $u = xz$. Since

$$u_i e_i \leqslant u \leqslant z \leqslant \bar{x}_i \cup (e_i \to y) \quad \text{and} \quad u_i e_i \leqslant e_i,$$

we have

$$u_i e_i \leqslant [\bar{x}_i \cup (e_i \to y)] e_i \leqslant \bar{x}_i e_i \cup y \quad \text{for } i = 1, 2, \ldots$$

Hence $u_i e_i x_i^{\urcorner} \leqslant y$ and, therefore, by Lemma 2.3,

$$u_i e_i = u_i e_i x = u_i e_i \bigcup_{j=1}^{\infty} x_j e_j = u_i \bigcup_{j=1}^{i} x_j e_j \leqslant \bigcup_{j=1}^{i} u_j x_j e_j \leqslant y \quad \text{for } i = 1, 2, \ldots$$

So we proved that $u = xz \leqslant y$.

Now suppose that $xw \leqslant y$ for a certain $w \in A$. Evidently, $x_i e_i w \leqslant xw \leqslant y$ for all i. Then $w \leqslant x_i e_i \to y$, i.e. (see [1], Lemma 2.6) $w \leqslant \bar{x}_i \cup (e_i \to y)$ for all i, and that is what we had to show: $w \leqslant y$.

In order to prove the converse implication, let us assume that $x \to y = z$ exists. We show that (3.1) holds. In fact,

$$x \to y \leqslant x_i e_i \to y = \bar{x}_i \cup (e_i \to y) \quad \text{for every } i.$$

On the other hand, the inequality $w \leqslant \bar{x}_i \cup (e_i \to y) = x_i e_i \to y$ for a certain $w \in A$ and every i yields $w x_i e_i \leqslant y$ for every i. Hence

$$w x e_j \leqslant w \bigcup_{i=1}^{j} x_i e_i \leqslant y, \quad \text{i.e., } w e_j \leqslant x \to y \text{ for } j = 1, 2, \ldots$$

Therefore, using a monotonic representation

$$w = \bigcup_{i=1}^{\infty} w_i e_i,$$

we get $w_i e_i \leqslant x \to y$ for all i and, consequently, $w \leqslant x \to y$. This completes the proof of (i).

Similarly, but using the dual representation (2.7), one can easily prove (ii).

LEMMA 3.2. *Let us assume that*

$$x = \bigcup_{i=1}^{\infty} x_i e_i \quad \textit{and} \quad y = \bigcup_{j=1}^{\infty} y_j e_j$$

are monotonic representations of $x \in A$ and $y \in A$, respectively.

(i) *Assume that $e_i \Rightarrow y$ exists for $i = 1, 2, \ldots$ Then $x \Rightarrow y$ exists if and only if the infinite meet*

$$z = \bigcap_{i=1}^{\infty}{}^{(B)} [\bar{x}_i \cup (e_i \Rightarrow y)]$$

exists in the center B of A. If they both exist, the identity $x \Rightarrow y = z$ holds.

(ii) *Assume that $x \Rightarrow e_i$ exists for $i = 0, 1, \ldots$ Then $x \Rightarrow y$ exists if and only if the infinite meet*

$$z = \bigcap_{i=1}^{\infty}{}^{(B)} [y_i \cup (x \Rightarrow e_{i-1})]$$

exists in B. If they both exist, the identity $x \Rightarrow y = z$ holds.

The proof is similar to that of Lemma 3.1 and will be omitted here.

Observe that part (i) of Lemma 3.1 has the following important consequence:

COROLLARY 3.1. *Assume that in A there exists the pseudo-complement $\neg e_i$ for $i = 1, 2, \ldots$ Then the pseudo-complement $\neg x$ exists for an $x \in A$ if and only if the infinite meet*

$$\bigcap_{i=1}^{\infty} (\bar{x}_i \cup \neg e_i)$$

exists. If both exist, then

$$\neg x = \bigcap_{i=1}^{\infty} (\bar{x}_i \cup \neg e_i). \tag{3.3}$$

LEMMA 3.3. *Let A be a P_0-lattice with the center B.*

(i) *If $e_i \to e_j$ exists for certain i, j, then*

$$e_i \to e_j = e_j \cup c_{ij}, \quad \textit{where} \quad c_{ij} \in B. \tag{3.4}$$

In particular, $\neg e_i \in B$, if it exists.

(ii) *If $e_i \Rightarrow e_j$ exists for a certain pair i, j, then $e_i \to e_j$ exists and*

$$e_i \to e_j = e_j \cup (\dot{e}_i \Rightarrow e_j). \tag{3.5}$$

Proof. (i) It is obvious for $i \leqslant j$. For $i > j$, let

$$e_i \to e_j = \bigcup_{k=1}^{\infty} c_{ij,k} e_k$$

be a monotonic representation. Then Lemma 2.3 yields

$$\bigcup_{k=1}^{i} c_{ij,k} e_k = (e_i \to e_j) e_i \leqslant e_j.$$

Therefore,

$$(e_j \cup c_{ij,i}) e_i = e_j \cup c_{ij,i} e_i \leqslant e_j, \quad \text{i.e.} \quad e_j \cup c_{ij,i} \leqslant e_i \to e_j.$$

On the other hand,

$$c_{ij,k} e_k \leqslant \begin{cases} e_j & \text{for } k \leqslant i, \\ c_{ij,i} & \text{for } k > i. \end{cases}$$

Thus $e_i \rightarrow e_j \leqslant e_j \cup c_{ij,i}$. In this way (3.4) has been proved for $c_{ij} = c_{ij,i}$. In order to prove (3.5) let us observe that

$$e_i[e_j \cup (e_i \Rightarrow e_j)] \leqslant e_j \quad \text{for } i > j.$$

If, on the other hand, a certain $x \in A$ has a monotonic representation

$$x = \bigcup_{k=1}^{\infty} x_k e_k$$

and $xe_i \leqslant e_j$ holds, then $x_k e_k \leqslant e_j$ for $k \leqslant i$. For $k > i$, we have $x_k \leqslant x_i \leqslant e_i \Rightarrow e_j$. Consequently,

$$x_k e_k \leqslant e_j \cup (e_i \Rightarrow e_j) \text{ for every } k, \quad \text{i.e., } x \leqslant e_j \cup (e_i \Rightarrow e_j).$$

In this way we proved (3.5).

THEOREM 3.1. *Let A be a P_0-lattice of order ω^+. Then*

(i) *A is a B-algebra if and only if it is a P-algebra.*

Let A be an FP_0-lattice. Then

(ii) *A is a Heyting algebra if and only if it is an L-algebra;*

(iii) *A is a Heyting algebra if and only if $e_i \rightarrow e_j$ exists for all i, j;*

(iv) *A is a B algebra if and only if $e_i \Rightarrow e_j$ exists for all i, j.*

Proof. The "only if" part of statements (i) and (ii) requires a proof. Let B be the center of A and let

$$x = \bigcup_{i=1}^{\infty} x_i e_i \quad \text{and} \quad y = \bigcup_{j=1}^{\infty} y_j e_j$$

be monotonic representations. By Lemma 3.2 (i) we conclude that

$$(*) \qquad x \Rightarrow y = \bigcap_{i=1}^{\infty}{}^{(B)}[\bar{x}_i \cup (e_i \Rightarrow y)], \qquad y \Rightarrow x = \bigcap_{j=1}^{\infty}{}^{(B)}[\bar{y}_j \cup (e_j \Rightarrow x)].$$

The inequalities

$$x_j \leqslant e_j \Rightarrow x \quad \text{and} \quad y_i \leqslant e_i \Rightarrow y$$

imply

$$(**) \qquad [\bar{x}_i \cup (e_i \Rightarrow y)] \cup [\bar{y}_j \cup (e_j \Rightarrow x)] \geqslant (\bar{x}_i \cup y_i) \cup (\bar{y}_j \cup x_j) = 1$$

for all positive indices i, j. In fact, $\bar{x}_i \cup x_j = 1$ for $i \geqslant j$, and $y_i \cup \bar{y}_j = 1$ for $i < j$. $(*)$ and $(**)$ easily imply that $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$, which proves (i).

We omit a similar proof of (ii).

In order to prove (iii) let us suppose that $e_i \rightarrow e_j$ exists for all i, j and that A is an FP_0-lattice. If

$$x = \bigcup_{i=1}^{\infty} x_i e_i \quad \text{and} \quad y = \bigcup_{j=1}^{\infty} y_j e_j$$

are monotonic representations, then, by Lemma 2.5 and Theorem 2.1, there exists a positive integer m (large enough) such that

$$x = \bigcup_{i=1}^{m} x_i e_i \cup x_\omega \quad \text{and} \quad y = \bigcap_{j=1}^{m} (y_j \cup e_{j-1}).$$

The remaining part of the proof easily follows from the fact, proved in [1], that

$$x_i e_i \to (y_j \cup e_k) = \bar{x}_i \cup y_j \cup (e_i \to e_k).$$

The proof of (iv) is similar.

Let A be a P_0-lattice. Then

$$x = \bigcup_{i=1}^{\infty} x_i e_i$$

is said to be the *highest monotonic representation* if $x'_i \leqslant x_i$ $(i = 1, 2, \ldots)$ for any monotonic representation

$$x = \bigcup_{i=1}^{\infty} x'_i e_i.$$

The *lowest monotonic representation* is defined analogously.

THEOREM 3.2. *If a P_0-lattice A of order ω^+ is a B-algebra, then each $x \in A$ has both the highest and the lowest monotonic representations, and*

(3.6) $\bigcup_{i=1}^{\infty} (e_i \Rightarrow x) e_i$ *is the highest representation of* x,

(3.7) $\bigcup_{i=1}^{\infty} \overline{(x \Rightarrow e_{i-1})} e_i$ *is the lowest representation of* x.

Proof. Let us consider a monotonic representation

$$x = \bigcup_{i=1}^{\infty} x_i e_i.$$

Observe that $x_i \leqslant e_i \Rightarrow x$ and $(e_i \Rightarrow x) e_i \leqslant x$. Therefore, if $(e_i \Rightarrow x) e_i \leqslant z$ for a certain $z \in A$ and for all i, then $x_i e_i \leqslant z$ for all i, as well; that is, $x \leqslant z$. So (3.6) is proved to be true for an arbitrary $x \in A$.

In order to prove (3.7) we will use the dual representation (see (2.7))

$$x = \bigcap_{i=1}^{\infty} (x_i \cup e_{i-1}).$$

Since $x \leqslant x_i \cup e_{i-1}$, we have

$$x \Rightarrow e_{i-1} \geqslant (x_i \cup e_{i-1}) \Rightarrow e_{i-1} \geqslant \bar{x}_i.$$

Then

$$(*) \qquad \overline{x \Rightarrow e_{i-1}} \leqslant x_i \quad \text{for all } i.$$

It remains to show that (see (2.7))

$$(**) \qquad x = \bigcap_{i=1}^{\infty} [(x \Rightarrow e_{i-1}) \cup e_{i-1}].$$

To achieve this let us observe first that

$$x = x(x \Rightarrow e_{i-1}) \cup x(\overline{x \Rightarrow e_{i-1}}) \leqslant e_{i-1} \cup (\overline{x \Rightarrow e_{i-1}}) \quad \text{for all } i.$$

On the other hand, if $z \leqslant (\overline{x \Rightarrow e_{i-1}}) \cup e_{i-1}$ for all i and a certain $z \in A$, then $z \leqslant x_i \cup e_{i-1}$ by $(*)$, i.e., $z \leqslant x$, and so $(**)$ is proved.

It is worth noticing and easy to see that the operator D_i defined by

$$D_i(x) = e_i \Rightarrow x \quad \text{for } i = 1, 2, \ldots$$

is a morphism in the category of bounded distributive lattices $D_{0,1}$. Using this notation, we can rewrite the highest monotonic representation of x in the form

$$x = \bigcup_{i=1}^{\infty} D_i(x) e_i$$

(for the operator D_i in Post algebras, see, e.g., [3]).

THEOREM 3.3. *Let A be a P_0-lattice of order ω^+, and let the center B of A be a σ-regular sublattice of A. Then A is a Stone lattice if and only if it is pseudo-complemented.*

Proof. Let A be pseudo-complemented and $x \in A$. Then, by Corollary 3.1,

$$\neg x = \bigcap_{i=1}^{\infty} (\bar{x}_i \cup \neg e_i)$$

for a monotonic representation

$$x = \bigcup_{i=1}^{\infty} x_i e_i.$$

But $\neg e_i$ is Boolean in virtue of Lemma 3.3. By the assumed regularity of the center B, $\neg x \in B$. Thus $\neg x \cup \neg\neg x = 1$, i.e., A is a Stone lattice. This completes the proof, since the converse implication is obvious.

If the center B is not σ-regular, the pseudo-complement $\neg x$ need not be Boolean. In this general case we have only the following theorem:

THEOREM 3.4. *Let A be a P_0-lattice of order ω^+ with the center B. Let us consider the following conditions:*

(1) *A is a Stone lattice;*

(2) *for every $x \in A$, there exists a monotonic representation $x = \bigcup_{i=1}^{\infty} b_i e_i$ such that $b_1 \leqslant c_1$ for any monotonic representation $x = \bigcup_{i=1}^{\infty} c_i e_i$;*

(3) *$x \Rightarrow 0$ exists for every $x \in A$;*

(4) *$\neg e_i$ exists for every i.*

Then (1) *implies* (2) *and* (2) *implies* (3), *and* (3) *and* (4) *are equivalent. If A is an FP_0-lattice, then each of the above conditions is equivalent to*

(5) *A is pseudo-complemented.*

Proof. (1) implies (2). Since $\neg x \cup \neg\neg x = 1$ for every x in the Stone lattice A, we have $\neg x \in B$, $\neg\neg x \in B$, and $\neg\neg x \geqslant x$. Let

$$x = \bigcup_{i=1}^{\infty} x_i e_i$$

be a monotonic representation. Then, by Lemma 2.1,

$$x = x \cap \neg\neg x = \bigcup_{i=1}^{\infty} (x_i \cap \neg\neg x) e_i.$$

By the same Lemma 2.1, $\bar{c}_1 x = 0$, i.e., $\bar{c}_1 \leqslant \neg x$. Hence

$$c_1 \geqslant \neg\neg x \geqslant x_1 \cap \neg\neg x = b_1.$$

(2) implies (3). Suppose that $xz = 0$ for a certain $z \in B$ and that (2) is satisfied. Then

$$x = x\bar{z} = \bigcup_{i=1}^{\infty} (\bar{z} b_i) e_i$$

is a monotonic representation. Hence $\bar{z} b_1 \geqslant b_1$ by (2), and this means that $z \leqslant \bar{b}_1$. Consequently, $\bar{b}_1 = x \Rightarrow 0$.

(3) implies (4). In virtue of Lemma 3.3, $\neg e_i$ exists and

$$\neg e_i = e_i \rightarrow e_0 = e_i \Rightarrow e_0.$$

(4) implies (3), almost trivially, by Lemma 3.3.

The remaining part of the proof will be omitted here. The reader may follow the lines of the proof of the similar Theorem 2.8 in [1].

REFERENCES

[1] G. Epstein and A. Horn, *Chain based lattices*, Pacific Journal of Mathematics 55 (1974), p. 65-84.

[2] T. Katriňák and A. Mitschke, *Stonesche Verbande der Ordnung n und Postalgebren*, Mathematische Annalen 199 (1972), p. 13-20.

[3] H. Rasiowa, *On generalized Post algebras of order ω^+ and ω^+-valued predicate calculi*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, astronomiques et physiques, 21 (1973), p. 209-219.

[4] — and R. Sikorski, *The mathematics of metamathematics*, Warszawa 1968.

[5] H. Sawicka, *On some properties of Post algebras with countable chain of constants*, Colloquium Mathematicum 25 (1972), p. 201-209.

[6] T. Traczyk, *Axioms and some properties of Post algebras*, ibidem 10 (1963), p. 193-209.

INSTITUTE OF MATHEMATICS
WARSAW TECHNICAL UNIVERSITY

Reçu par la Rédaction le 25. 7. 1974