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THE IDEAL STRUCTURE OF SIMPLE TERNARY ALGEBRAS

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1. Introduction. Let V be a non-empty set and Q a ternary operation defined on V. The pair $(V, Q) = A$ is called a *simple ternary algebra* if Q satisfies the following conditions:

(1) $Q(a, a, b) = a,\ a, b \in V$;

(2) $Q(a, b, c)$ is invariant under all 6 permutations of $a, b, c \in V$;

(3) $Q(Q(a, b, c), d, e) = Q(Q(a, d, e), Q(b, d, e), c),\ a, b, c, d, e \in V$.

Simple ternary algebras are considered by Avann [1], Sholander [10]-[12], Nebeský [5]-[7], and Nieminen [8], [9]. In [5], Nebeský defined the ideal concept for a class of simple ternary algebras. The purpose of this paper is to prove a few properties of the ideals of simple ternary algebras. These properties are analogous to those of the ideals of distributive lattices (see, e.g., [3], Section 7). The results obtained here are also generalizations of the filter structure of prime semilattices determined by Balbes [2] and of the structure of the convex sublattices of distributive lattices given by Koh [4].

Each distributive lattice is a simple ternary algebra when we define

$$Q(a, b, c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c),$$

but Theorem 3.4 in [2] shows that not every property of distributive lattices can be proved for simple ternary algebras.

2. The lattice of ideals. Let $A = (V, Q)$ be a simple ternary algebra and let U and W be two non-empty subsets of V. For any $s \in V$ we write

$$Q(U, W, s) = \bigcup \{Q(u, w, s) \mid u \in U \text{ and } w \in W\}.$$

We shall use the definition given by Nebeský [5]: A non-empty subset W of V is called an *ideal of the algebra* $A = (V, Q)$ whenever $Q(W, W, z) \subseteq W$ for each $z \in V$. The converse inclusion is obvious in virtue of (1), and thus $W = Q(W, W, z)$ for each $z \in V$ if and only if W is an ideal of A. Furthermore, an ideal P is called *prime* if $Q(x, y, z) \in P$ implies $|P \cap \{x, y, z\}| \geqslant 2$, where $|B|$ denotes the cardinality of the set B.

Avann [1], Lemma 3, has shown the following useful connection between simple ternary algebras and meet-semilattices. Let $A = (V, Q)$ be a simple ternary algebra and let $x \in V$. Then A can be associated with a partial lattice $L(A, x)$ by the following properties:

(4) The order relation in $L(A, x)$ is given by

$$b \leqslant c \Leftrightarrow Q(x, b, c) = b.$$

(5) The zero element of $L(A, x)$ is x.

(6) $L(A, x)$ is closed with respect to the meet given by $b \wedge c = Q(x, b, c)$.

(7) The existence of an element m $(b, c \leqslant m)$ implies the existence of the join $b \vee c = Q(m, b, c)$.

(8) If $b \vee c$ exists, then $d \wedge (b \vee c) = (d \wedge b) \vee (d \wedge c)$, $d \in V$.

(9) For all triples $b, c, d \in V$ there exists

$$(b \wedge c) \vee (b \wedge d) \vee (c \wedge d) = Q(b, c, d).$$

Let W be an ideal of a simple ternary algebra $A = (V, Q)$ and let $x \in V \setminus W$. In the partial lattice $L(A, x)$, $Q(a, b, x) = a \wedge b \in W$ for each two elements $a, b \in W$, and if in $L(A, x)$ there is an element $m \geqslant a, b$, then also $Q(a, b, m) = a \vee b \in W$. Moreover, if $a \leqslant y \leqslant b$, then $Q(a, b, y) = y$ according to (7), and if $Q(a, b, y) \in W$, then also $y \in W$. Hence the ideal W of A is a convex subset of the partial lattice $L(A, x)$. The definition of an ideal in A implies now the following obvious characterization:

LEMMA 1. *A non-empty subset $S \subseteq V$ is an ideal of the simple ternary algebra $A = (V, Q)$ if and only if $s \wedge t \in S$ for each $L(A, x)$ with $x \in V \setminus S$ and for each two elements $s, t \in S$.*

If we treat the empty set as the least ideal of the algebra $A = (V, Q)$, then we can construct the lattice $\mathscr{I}(A)$ of ideals of A. This is done in the following lemma:

LEMMA 2. *Let W and U be two ideals of a simple ternary algebra $A = (V, Q)$. Then $W \cap U$ is an ideal of A, and*

$$Z = \{z \mid Q(u, w, z) = z \text{ for some } u \in U \text{ and some } w \in W\}$$

is the least ideal of A containing U and W.

Proof. If $z, y \in U \cap W$, then $Q(z, y, s) \in U, W$ for each $s \in V$ according to the definition of an ideal in A, and so $U \cap W$ is an ideal of A.

We show that Z is an ideal of A. Let $q \in V \setminus Z$ and $x, y \in Z$. If we write $s = Q(x, y, q)$, then $s = Q(x, y, s)$ by (3). Let us assume that $s \notin Z$. Then $x \wedge y = s$ in the partial lattice $L(A, s)$. Since $x \in Z$, we have $x = Q(u_x, w_x, x)$ with $u_x \in U$ and $w_x \in W$. In $L(A, s)$ this means that

$$x = (x \wedge w_x) \vee (x \wedge u_x) \vee (w_x \wedge u_x),$$

whence $x \geqslant u_x \wedge w_x$. Similarly, $y \geqslant w_y \wedge u_y$ in $L(A, s)$, where $y = Q(w_y, u_y, y)$, $u_y \in U$ and $w_y \in W$. But

$$s = x \wedge y \geqslant (w_x \wedge u_x) \wedge (w_y \wedge u_y),$$

and so $(w_x \wedge u_x) \wedge (w_y \wedge u_y) = s$ in $L(A, s)$. On the other hand,

$$s = x \wedge y = (w_x \wedge w_y) \wedge (u_x \wedge u_y) = Q\big(Q(w_x, w_y, s), Q(u_x, u_y, s), s\big),$$

where $Q(w_x, w_y, s) \in W$ and $Q(u_x, u_y, s) \in U$. Hence $s \in Z$, which is a contradiction.

Let $K \subseteq V$ be an ideal of A containing W and U. Then $Q(u, w, p) \in K$ for each two elements $u \in U$, $w \in W$ and for each $p \in V$. But then $Z \subseteq K$, and so Z is the least ideal of A containing U and W. This completes the proof.

The following theorem is a generalization of the property of relatively maximal ideals of distributive lattices (see, e.g., [3], Section 7, Theorem 15, and [2], Theorem 2.2).

THEOREM 1. *Let $A = (V, Q)$ be a simple ternary algebra and let J and I be two non-empty disjoint ideals of A. Then there exists a prime ideal P of A such that $I \subseteq P$ and $P \cap J = \emptyset$. Moreover, let $A' = (V', Q)$ be a ternary algebra satisfying conditions* (1) *and* (2), *and such that for any pair J', I' of two non-empty disjoint ideals of A' there exists a prime ideal P' of A' with $I' \subseteq P'$ and $P' \cap J' = \emptyset$. Then A' is a simple ternary algebra.*

Proof. 1° We show first the latter part of the theorem. Assume that

$$Q\big(Q(x, y, z), u, w\big) \neq Q\big(Q(x, u, w), Q(y, u, w), z\big) \quad \text{in } A'.$$

We put

$$J' = \big\{Q\big(Q(x, u, w), Q(y, u, w), z\big)\big\} \quad \text{and} \quad I' = \big\{Q\big(Q(x, y, z), u, w\big)\big\}.$$

According to (1), J' and I' are non-empty ideals of A'. As $I' \subseteq P'$,

$$|P' \cap \{Q(x, y, z), u, w\}| \geqslant 2.$$

If $u, w \in P'$, then $Q(x, u, w), Q(y, u, w) \in P'$ and, consequently, $J' \subseteq P'$, which is a contradiction. The proof goes along the same lines if $Q(x, y, z), u \in P'$ (or $Q(x, y, z), w \in P'$). Hence A' is a simple ternary algebra.

2° Let $j \in J$. We consider the partial lattice $L(A, j)$. As J and I are convex sets of $L(A, j)$ and j is the least element of $L(A, j)$, there is no element $j' \in J$ such that $j' > i$ in $L(A, j)$ for some $i \in I$. We define now a set M in $L(A, j)$ as follows:

$$M = \{m \mid m \geqslant i \text{ for some } i \in I,\ m \in V\}.$$

If $a \geqslant b$ and $b \in M$, then, obviously, $a \in M$. Furthermore, if $a, b \in M$, then $a \wedge b \geqslant i_a \wedge i_b$, where $a \geqslant i_a \in I$ and $b \geqslant i_b \in I$. Hence M is a filter of $L(A, j)$ and $M \cap J = \emptyset$. But now we can apply Balbes' Theorem 2.2 to M and J, which shows that there is a prime filter P of the partial lattice $L(A, j)$ such that $M \subseteq P$ and $P \cap J = \emptyset$. We show now that P is also a prime ideal of A.

If $x_1, x_2 \in P$, then

$$Q(x_1, x_2, s) = (x_1 \wedge x_2) \vee (x_1 \wedge s) \vee (x_2 \wedge s)$$

implies

$$Q(x_1, x_2, s) \geqslant x_1 \wedge x_2 \quad \text{for each } s \in V.$$

As $x_1 \wedge x_2 \in P$, we have $Q(x_1, x_2, s) \in P$ for each $s \in V$. Hence P is an ideal of A. Moreover, if $Q(x, y, z) \in P$, then the relation

$$(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \in P$$

implies that at least one of the elements $x \wedge y$, $x \wedge z$, and $y \wedge z$ belongs to P, as P is a prime filter of $L(A, j)$. If $x \wedge y \in P$, then $x, y \in P$, as $x, y \geqslant x \wedge y$. But this shows that $Q(x, y, z) \in P$ implies $|P \cap \{x, y, z\}| \geqslant 2$. Thus P is a prime ideal of the algebra A, and the theorem follows.

Note that although every prime filter of $L(A, j)$ is a prime ideal of A, the converse need not hold; this can easily be seen by means of trees.

The following theorem is also a generalization of the corresponding results for distributive lattices (see [3], Section 7, Corollary 18).

THEOREM 2. *A ternary algebra A satisfying conditions* (1) *and* (2) *is simple if and only if every ideal of A is the meet of the prime ideals which contain it.*

Proof. Suppose that A is simple, and let U be an ideal of A such that

$$U \neq \bigcap_r P_r$$

for all prime ideals P_r containing U. Then there exists an element

$$x \in \bigcap_r P_r \setminus U.$$

$\{x\}$ is an ideal of A and, according to Theorem 1, there exists a prime ideal P of A such that $U \subseteq P$ and $\{x\} \cap P = \emptyset$. This is a contradiction, and then every ideal of A is the meet of the prime ideals which contain it.

The converse proof is similar to that of the latter part of Theorem 1.

THEOREM 3. *In the lattice $\mathscr{I}(A)$ every subset*

$$[X) = \{Y \mid Y \geqslant X,\ Y, X \in \mathscr{I}(A)\}$$

is a distributive sublattice of $\mathscr{I}(A)$.

Proof. Let $Z, U, W \in [X)$. As it is well known, the distributivity of $[X)$ follows from the relation

$$Z \wedge (U \vee W) \subseteq (Z \wedge U) \vee (Z \wedge W).$$

Let $k \in Z \wedge (U \vee W)$. As $k \in U \vee W$, we have $k = Q(u, w, k)$, where $u \in U$ and $w \in W$. We consider the partial lattice $L(A, k)$. For each $t \in Z$ and for each $s \in V$, we have $s \wedge t \in Z$ as $s \wedge t = Q(s, z, k)$ and $z, k \in Z$. If $z \in U \wedge W \wedge Z$, then

$$Q(u, w, k) = u \wedge w = k$$

and, further, $z \wedge u \wedge w = k$ as k is the least element of $L(A, k)$. Thus

$$k = (z \wedge u) \wedge (z \wedge w) = Q\big(Q(z, u, k), Q(z, w, k), k\big).$$

If $z, k \in Z$, then $Q(z, u, k) \in Z$, and if $z, u \in U$, then also $Q(z, u, k) \in U$. Similarly, $Q(z, w, k) \in Z \wedge W$. According to Lemma 2,

$$Q\big(Q(z, u, k), Q(z, w, k), k\big) \in (Z \wedge U) \vee (Z \wedge W),$$

and the theorem follows.

3. The ternary algebra of ideals. In this section we consider the ternary algebra of ideals of a ternary algebra $A = (V, Q)$. We prove the analogue of the well-known theorem concerning the distributivity of the ideal lattice of a lattice.

THEOREM 4. *The ternary algebra $\mathscr{W}(A)$ of non-empty ideals of a simple ternary algebra is simple.*

Proof. Let U, W, K be three non-empty ideals of the ternary algebra $A = (V, Q)$. At first, we show that

$$Q(U, W, K) = \bigcup \{Q(u, w, k) \mid u \in U, w \in W \text{ and } k \in K\}$$

is an ideal of A.

Clearly, $Q(U, W, K)$ is a non-empty subset of V. Let $z, y \in Q(U, W, K)$ and consider an element $t = Q(z, y, f) = Q(z, y, t)$, where $f \in V$. We consider the partial lattice $L(A, t)$. As $z, y \in Q(U, W, K)$, we have

$$z = Q(u_z, w_z, k_z) \quad \text{and} \quad y = Q(u_y, w_y, k_y),$$

where $u_z, u_y \in U$, $w_z, w_y \in W$, and $k_z, k_y \in K$. Now

$$\begin{aligned} t &= Q(z, y, t) \\ &= [(u_z \wedge w_z) \vee (u_z \wedge k_z) \vee (w_z \wedge k_z)] \wedge [(u_y \wedge w_y) \vee (u_y \wedge k_y) \vee (k_y \wedge w_y)] \end{aligned}$$

and, applying property (8), we obtain

$$t = (u_z \wedge w_z \wedge u_y \wedge w_y) \vee (u_z \wedge k_z \wedge u_y \wedge k_y) \vee (w_z \wedge k_z \wedge w_y \wedge k_y) \vee \\ \vee (u_z \wedge w_z \wedge u_y \wedge k_y) \vee \ldots$$

As t is the least element of $L(A, t)$, already the join of the first three meets written above is equal to t. Hence we can write

$$t = [(u_z \wedge u_y) \wedge (w_z \wedge w_y)] \vee [(u_z \wedge u_y) \wedge (k_z \wedge k_y)] \vee [(w_z \wedge w_y) \wedge (k_z \wedge k_y)]$$
$$= Q\big(Q(u_z, u_y, t), Q(w_z, w_y, t), Q(k_z, k_y, t)\big),$$

where $Q(u_z, u_y, t) \in U$, $Q(w_z, w_y, t) \in W$, and $Q(k_z, k_y, t) \in K$. Thus $Q(U, W, K)$ is a non-empty ideal of A.

In $\mathscr{W}(A) = \big(\mathscr{I}(A)\setminus\emptyset, Q\big)$ the operation Q satisfies conditions (1) and (2). As A is simple,

$$Q\big(Q(U, W, K), S, T\big) \subseteq Q\big(Q(U, S, T), Q(W, S, T), K\big),$$

whenever U, W, K, S, and T are non-empty ideals of A. Let

$$g \in Q\big(Q(U, S, T), Q(W, S, T), K\big).$$

Then the following considerations are performed in the partial lattice $L(A, g)$:

$$g = Q\big(Q(u, s_1, t_1), Q(w, s_2, t_2), k\big)$$
$$= \{[(u \wedge s_1) \vee (u \wedge t_1) \vee (s_1 \wedge t_1)] \wedge [(w \wedge s_2) \vee (w \wedge t_2) \vee (s_2 \wedge t_2)]\} \vee$$
$$\vee (k \wedge u \wedge s_1) \vee (k \wedge u \wedge t_1) \vee (k \wedge s_1 \wedge t_1) \vee (k \wedge w \wedge t_2) \vee (k \wedge w \wedge s_2) \vee (k \wedge s_2 \wedge t_2).$$

As g is the least element in $L(A, g)$, each term of a join equal to g is also equal to g, and $g \wedge r = g$ for each $r \in V$. Using these observations and developing the term in { }-brackets above, we obtain

$$g = (s_1 \wedge s_2 \wedge t_1 \wedge t_2) \vee (u \wedge w \wedge s_1 \wedge s_2) \vee (u \wedge k \wedge s_1 \wedge s_2) \vee$$
$$\vee (w \wedge k \wedge s_1 \wedge s_2) \vee (u \wedge w \wedge t_1 \wedge t_2) \vee (u \wedge k \wedge t_1 \wedge t_2) \vee (w \wedge k \wedge t_1 \wedge t_2)$$
$$= [(s_1 \wedge s_2) \wedge (t_1 \wedge t_2)] \vee \{(s_1 \wedge s_2) \wedge [(u \wedge w) \vee (u \wedge k) \vee (w \wedge k)]\} \vee$$
$$\vee \{(t_1 \wedge t_2) \wedge [(u \wedge w) \vee (u \wedge k) \vee (w \wedge k)]\}$$
$$= Q\big((u \wedge w) \vee (u \wedge k) \vee (w \wedge k), s_1 \wedge s_2, t_1 \wedge t_2\big)$$
$$= Q\big(Q(u, w, k), Q(s_1, s_2, g), Q(t_1, t_2, g)\big),$$

where $Q(s_1, s_2, g) \in S$ and $Q(t_1, t_2, g) \in T$. Hence also

$$Q\big(Q(U, S, T), Q(W, U, T), K\big) \subseteq Q\big(Q(U, W, K), S, T\big),$$

and so (3) holds in $\mathscr{W}(A) = \big(\mathscr{I}(A)\setminus\emptyset, Q\big)$.

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