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COMPACT CONVOLUTION OPERATORS BETWEEN $L_p(G)$-SPACES

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1. Let G be a unimodular locally compact Hausdorff group, and $L_p = L_p(G)$ $(1 \leqslant p \leqslant \infty)$ the usual corresponding spaces with respect to (left and right) Haar measure dy. For a complex-valued function g on G and $a \in G$, the left and right translates of g are defined by

$$({}_ag)(x) = g(a^{-1}x) \quad \text{and} \quad (g_a)(x) = g(xa^{-1}),$$

respectively.

Let θ be an element of L_p. The element θ is called ***left almost periodic*** (l.a.p.) if the set $\{{}_a\theta\colon a \in G\}$ of left translates is relatively compact (i.e. totally bounded) in the space L_p with its usual L_p-norm. An analogous definition is valid for a *right almost periodic* (r.a.p.) element of L_p.

The space L_p with the convolution mapping $(f, \theta) \to f * \theta$ of $L_1 \times L_p$ into L_p is a left Banach L_1-module. In particular, a fixed $\theta \in L_p$ induces a linear operator H_θ from L_1 into L_p by means of

$$[H_\theta(f)](x) = (f * \theta)(x) = \int_G f(xy^{-1})\theta(y)\,dy \qquad (f \in L_1).$$

Put $b(L_1) = \{f \in L_1\colon \|f\|_1 \leqslant 1\}$. Then the operator H_θ is said to be *compact* if $H_\theta\big(b(L_1)\big)$ is relatively compact in the Banach space $(L_p, \|\cdot\|_p)$.

2. We want to prove the following theorem:

THEOREM. *H_θ is compact iff θ is left almost periodic.*

In the proof we will use the following lemmas:

LEMMA 1. *If $\{\theta_\lambda\}_{\lambda \in \Lambda}$ is a net in L_p consisting of l.a.p. functions, and $\{\theta_\lambda\}_{\lambda \in \Lambda}$ converges in the L_p-norm to $\theta \in L_p$, then the limit θ is also l.a.p.*

The easy proof is omitted.

LEMMA 2. *If G is compact, then every θ in L_p $(1 \leqslant p < \infty)$ is l.a.p.*

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This follows from the fact that, for fixed θ in L_p, the mapping $x \to {}_x\theta$ of G into L_p is continuous.

LEMMA 3. *If G is not compact and θ is l.a.p. in L_p $(1 \leqslant p < \infty)$, then θ is the zero element of L_p.*

Proof. Let $\theta \in L_p$ be left almost periodic and let $\varepsilon > 0$ be given. There exist a finite number of points $a_1, a_2, \ldots, a_n$ in G such that, for each $b \in G$, a point a_i $(i \in \{1, 2, \ldots, n\})$ may be chosen for which

$$\|{}_b\theta - {}_{a_i}\theta\|_p < \varepsilon. \tag{1}$$

K being a compact set in G, the set $\big(\bigcup_{i=1}^{n} a_i K\big)K^{-1}$ is compact. By assumption, a point $a \in G$ may be found such that

$$a \notin \Big(\bigcup_{i=1}^{n} a_i K\Big)K^{-1};$$

this also means that

$$(aK) \cap \Big(\bigcup_{i=1}^{n} a_i K\Big) = \emptyset.$$

From (1) it follows that to this element a there corresponds a point a_j $(j \in \{1, 2, \ldots, n\})$ such that

$$\|({}_a\theta - {}_{a_j}\theta)\chi_{aK}\|_p < \varepsilon$$

(χ_{aK} denotes the characteristic function of aK). Using this result we obtain

$$\begin{aligned} \|\theta\|_p = \|{}_a\theta\|_p &\leqslant \|{}_a\theta\chi_{aK}\|_p + \|{}_a\theta\chi_{G\setminus aK}\|_p \leqslant \varepsilon + \|{}_{a_j}\theta\chi_{aK}\|_p + \|{}_a\theta\chi_{G\setminus aK}\|_p \\ &\leqslant \varepsilon + \|{}_{a_j}\theta\chi_{G\setminus a_jK}\|_p + \|{}_a\theta\chi_{G\setminus aK}\|_p. \end{aligned} \tag{2}$$

But

$$\|{}_a\theta\chi_{G\setminus aK}\|_p = \|{}_{a^{-1}}({}_a\theta\chi_{G\setminus aK})\|_p,$$

and the function in the second member is equal to $\theta\chi_{G\setminus K}$. The same may be done for ${}_{a_j}\theta\chi_{G\setminus a_jK}$. Hence from (2) it follows that

$$\|\theta\|_p \leqslant \varepsilon + 2\|\theta\chi_{G\setminus K}\|_p. \tag{3}$$

Choosing finally the compact set K such that

$$\int_{G\setminus K} |\theta(x)^p|\, dx < \varepsilon^p$$

we obtain $\|\theta\|_p < 3\varepsilon$ from (3). Hence $\theta = 0$ in L_p.

Proof of the Theorem. Suppose that H_θ is compact. If $f \in b(L_1)$ and $a \in G$, then $\|{}_af\|_1 = \|f\|_1$ and ${}_a(f * \theta) = {}_af * \theta$. So

$$\{{}_a(f*\theta):\ a \in G\} = \{{}_af*\theta:\ a \in G\} \subset \{g*\theta:\ g \in b(L_1)\},$$

from which we conclude that $f*\theta$ is l.a.p. for each $f \in b(L_1)$. Since

$$\frac{f}{\|f\|_1} \in b(L_1) \quad \text{for } f \in L_1,$$

$f*\theta$ is l.a.p. for each $f \in L_1$.

In L_1 there exists a symmetric approximate unit; this means that there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ in L_1 such that $e_\lambda(x^{-1}) = e_\lambda(x)$ for all $\lambda \in \Lambda$ and $x \in G$, $\|e_\lambda\|_1 = 1$, and

$$\lim_\lambda \|e_\lambda * f - f\|_1 = \lim_\lambda \|f*e_\lambda - f\|_1 = 0 \quad \text{for each } f \in L_1.$$

For $1 \leqslant p < \infty$, it is known (see [3], p. 273) that

$$\lim_\lambda \|e_\lambda * \theta - \theta\|_p = 0.$$

Since each $e_\lambda * \theta$ is l.a.p., it follows from Lemma 1 that so is θ.

For $p = \infty$ it follows easily, using the equality

$$\int_G f(x)(e_\lambda * \theta)(x)\,dx = \int_G \theta(x)(e_\lambda * f)(x)\,dx \quad (f \in L_1),$$

that $\{e_\lambda * \theta\}_{\lambda\in\Lambda}$ converges to θ for the w*-topology of L_∞. Hence a subnet of $\{e_\lambda * \theta\}_{\lambda\in\Lambda}$ converges to θ in $(L_\infty, \|\cdot\|_\infty)$. Using again Lemma 1, we conclude that θ is l.a.p.

To prove the converse, we first suppose that θ is l.a.p. in L_∞. Let $\varepsilon > 0$ be given. Since θ is also r.a.p., there exist a finite number of points $a_1, a_2, \ldots, a_n$ in G such that, for each $a \in G$, an element a_i $(i \in \{1, 2, \ldots, n\})$ may be found for which

$$\|\theta_{a^{-1}} - \theta_{a_i^{-1}}\|_\infty < \frac{\varepsilon}{3}. \tag{4}$$

For each $a \in G$ and each $f \in b(L_1)$ we have $|(f*\theta)(a)| \leqslant \|\theta\|_\infty$. Hence the set $\{(f*\theta)(a_1), (f*\theta)(a_2), \ldots, (f*\theta)(a_n)\}$, where f runs over $b(L_1)$, is a totally bounded subset of C^n (i.e. the n-dimensional complex space). So there exist a finite number of functions $f_1, f_2, \ldots, f_m$ in $b(L_1)$ such that, for each $f \in b(L_1)$, an element f_j $(j \in \{1, 2, \ldots, m\})$ may be found for which

$$|(f*\theta)(a_i) - (f_j*\theta)(a_i)| < \frac{\varepsilon}{3} \quad \text{for all } i \in \{1, 2, \ldots, n\}. \tag{5}$$

From (4) and (5) we easily see that

$$|(f*\theta)(a) - (f_j*\theta)(a)| < \varepsilon.$$

Hence $\|f*\theta - f_j*\theta\|_\infty < \varepsilon$, which means that the set $H_\theta(b(L_1))$ is relatively compact; so H_θ is a compact operator.

In view of Lemmas 2 and 3 the proof of the converse implication will be completed if we show that, for compact G, each θ in L_p $(1 \leqslant p < \infty)$ induces a compact H_θ. Using the results and notation of [3], § 27, we denote by $\mathscr{U}(G)$ the set of all continuous irreducible unitary representations of G, by Σ the set of equivalence classes of the representations, $\sigma \in \Sigma$, and by $U^{(\sigma)}$ a representation in the set σ. If $u_{jk}^{(\sigma)}$ is a coordinate function for $U^{(\sigma)}$, then it belongs to L_p. For $f \in b(L_1)$ we have

$$(6) \qquad (f * u_{jk}^{(\sigma)})(x) = \int_G f(y) u_{jk}^{(\sigma)}(y^{-1}x)\,dy = \sum_{r=1}^{d_\sigma} u_{rk}^{(\sigma)}(x) \Big(\int_G f(y) u_{jr}^{(\sigma)}(y^{-1})\,dy \Big),$$

where d_σ denotes the dimension of the representation space H_σ of $U^{(\sigma)}$. Since

$$\Big| \int_G f(y) u_{jr}^{(\sigma)}(y^{-1})\,dy \Big|$$

is bounded for all $f \in b(L_1)$, for given $\varepsilon > 0$ there exists a finite number of elements $f_1, f_2, \ldots, f_n$ of $b(L_1)$ such that, for each $f \in b(L_1)$, an f_i $(i \in \{1, 2, \ldots, n\})$ may be found for which

$$\Big| \int_G f(y) u_{jr}^{(\sigma)}(y^{-1})\,dy - \int_G f_i(y) u_{jr}^{(\sigma)}(y^{-1})\,dy \Big| < \varepsilon.$$

It follows that each operator, defined by each term in the last member of (6), is compact. Hence every $u_{jk}^{(\sigma)}$ induces a compact operator in L_p, and so do their linear combinations, which form a dense set T in $C(G)$, and hence in L_p (when σ runs over Σ). Since the operator norm of H_θ does not exceed $\|\theta\|_p$, T is dense in L_p also in the operator norm. Hence every operator H_θ is compact as a uniform limit of compact operators.

Denoting by AP the set of almost periodic elements of L_∞, we obtain from the foregoing the following result:

COROLLARY 1. *Let $\theta \in \mathrm{AP}$; then H_θ is compact, and thus $f * \theta \in \mathrm{AP}$ $(f \in L_1)$.*

This means that $L_1 * \mathrm{AP} \subset \mathrm{AP}$. The closure in $(L_\infty, \|\cdot\|_\infty)$ of $L_1 * \mathrm{AP}$ is equal to AP, as seen in the first part of the proof of the Theorem. But, by [3], p. 268, $L_1 * \mathrm{AP}$ is norm closed in AP. Hence $L_1 * \mathrm{AP} = \mathrm{AP}$.

In an Abelian locally compact Hausdorff group G, each multiplier $T\colon L_1(G) \to L_\infty(G)$ is L_∞-induced (see [4], p. 68). From the Theorem we conclude that

COROLLARY 2. *A multiplier $T\colon L_1(G) \to L_\infty(G)$ is compact iff it is induced by an almost periodic function.*

Remark. For $p = \infty$ the Theorem gives an extension of a part of a proposition by de Vito [1].

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