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ON SOME RIEMANNIAN MANIFOLDS ADMITTING A METRIC SEMI-SYMMETRIC CONNECTION

BY

J. KRAWCZYK (OPOLE)

1. Introduction. In a Riemannian manifold M ($\dim M = n \geqslant 4$) with a (possibly indefinite) metric g_{ij} we can consider a metric semi-symmetric connection with connection coefficients $\mathring{\Gamma}^h_{ji}$ (see [3]) given by

$$\mathring{\Gamma}^h_{ji} = \Gamma^h_{ji} + \delta^h_j p_i - p^h g_{ji}, \tag{1}$$

where Γ^h_{ji} are Christoffel symbols of M, p_i is a gradient vector, and $p^h = g^{hr} p_r$. Throughout the paper, the operator ∇_j means covariant differentiation with respect to the Riemannian connection, and the operator $\mathring{\nabla}_j$ means covariant differentiation with respect to the metric semi-symmetric connection (1). The curvature tensor $\mathring{R}^h{}_{ijk}$ of $\mathring{\Gamma}^h_{ji}$ and $R^h{}_{ijk}$ of Γ^h_{ji} are related by (see [2])

$$\mathring{R}_{hijk} = R_{hijk} - \alpha_{ij} g_{hk} + \alpha_{hj} g_{ik} - \alpha_{hk} g_{ij} + \alpha_{ik} g_{hj},$$

where α_{ij} is a tensor field of type (0, 2) defined by $\alpha_{ij} = \nabla_i p_j - p_i p_j + \frac{1}{2} p_r p^r g_{ij}$ and $\mathring{R}_{hijk} = g_{hr} \mathring{R}^r{}_{ijk}$. Let $\mathring{C}_{hijk}$ denote the conformal curvature tensor relative to the metric semi-symmetric connection, i.e.,

$$\mathring{C}_{hijk} = \mathring{R}_{hijk} - \frac{1}{n-2}(g_{ij}\mathring{R}_{hk} - g_{ik}\mathring{R}_{hj} + g_{hk}\mathring{R}_{ij} - g_{hj}\mathring{R}_{ik}) + \frac{\mathring{R}}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}),$$

where $\mathring{R}_{hk} = \mathring{R}_{hijk} g^{ij}$ and $\mathring{R} = \mathring{R}_{hk} g^{hk}$.

Adati and Miyazawa introduced [1] the concept of a *conformally recurrent manifold*. It is defined as an n-dimensional ($n > 3$) Riemannian manifold whose Weyl conformal curvature tensor

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}) + \frac{R}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj})$$

satisfies $\nabla_l C_{hijk} = a_l C_{hijk}$ for some vector field a_l.

An n-dimensional ($n > 3$) Riemannian manifold (M, g) is called *conformally recurrent with respect to* $\mathring{\nabla}$ if the conformal curvature tensor $\mathring{C}_{hijk}$ satisfies the condition

$$\mathring{\nabla}_l \mathring{C}_{hijk} = \mathring{a}_l \mathring{C}_{hijk} \tag{2}$$

for some vector field $\mathring{a}_l$. If $\mathring{\nabla}_l \mathring{C}_{hijk} = 0$ everywhere on M and $\dim M \geqslant 4$, then (M, g) is said to be *conformally symmetric with respect to* $\mathring{\nabla}$.

The purpose of this paper is to investigate conformally recurrent manifolds with respect to $\mathring{\nabla}$. Namely, we shall find (Theorem 3) some metric properties of a Riemannian manifold which is simultaneously conformally recurrent with respect to $\mathring{\nabla}$ and ∇.

All manifolds under consideration are assumed to be connected and of class C^∞. Their Riemannian metrics are not assumed to be definite.

2. Preliminaries. In the sequel we shall need the following lemmas:

LEMMA 1. *The Weyl conformal curvature tensor satisfies the well-known relations*

$$C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi},$$

$$C_{hijk} + C_{hkij} + C_{hjki} = 0, \qquad C^r{}_{ijr} = C^r{}_{irj} = C^r{}_{rij} = 0,$$

$$\nabla_l C_{hijk} + \nabla_j C_{hikl} + \nabla_k C_{hilj} = \frac{1}{n-3}(g_{ik}C_{hjl} + g_{hj}C_{ikl} + g_{il}C_{hkj} + g_{hk}C_{ilj} + g_{ij}C_{hlk} + g_{hl}C_{ijk}), \tag{3}$$

where $C_{ijk} = \nabla_r C^r{}_{ijk}$.

LEMMA 2 ([4], Lemma 3). *If* c_j, p_j *and* B_{hijk} *are numbers satisfying*

$$c_l B_{hijk} + p_h B_{lijk} + p_i B_{hljk} + p_j B_{hilk} + p_k B_{hijl} = 0,$$

$$B_{hijk} + B_{hjki} + B_{hkij} = 0, \qquad B_{hijk} = B_{jkhi} = -B_{hikj},$$

then each $b_j = c_j + 2p_j$ *is zero or each* B_{hijk} *is zero.*

LEMMA 3 ([2], Proposition 3.1). *The conformal curvature tensor relative to the metric semi-symmetric connection is equal to the conformal curvature tensor relative to the Riemannian connection.*

3. Main results.

PROPOSITION 1. *Suppose that M admits a metric semi-symmetric connection* (1) *such that*

$$\mathring{\nabla}_l \mathring{C}_{kjih} = 0. \tag{4}$$

Then

$$\nabla_l C_{kjih} + \nabla_j C_{lkih} + \nabla_k C_{jlih} = (1-n)(p_l C_{kjih} + p_j C_{lkih} + p_k C_{jlih}).$$

Proof. Differentiating $\mathring{C}_{kjih}$ covariantly and using (1), (4) and Lemma 3, we get

$$\nabla_l C_{kjih} = p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil} - (g_{lk} B_{jih} + g_{jl} B_{khi} + g_{il} B_{hkj} + g_{hl} B_{ijk}), \tag{5}$$

where $B_{jih} = p_r C^r{}_{jih}$. The last result, by Lemma 1, (5), and contraction with g^{hl}, implies $C_{ijk} = (1-n) B_{ijk}$, which together with (3) yields

$$\nabla_l C_{kjih} + \nabla_j C_{lkih} + \nabla_k C_{jlih} = \frac{1-n}{n-3}(g_{ik} B_{hjl} + g_{hj} B_{ikl} + g_{il} B_{hkj} + g_{hk} B_{ilj} + g_{ij} B_{hlk} + g_{hl} B_{ijk}). \tag{6}$$

Permuting in (5) the indices l, k, j cyclically, adding the resulting equations to (5) and using (6), we obtain

$$(3-n)(p_j C_{klih} + p_l C_{jkih} + p_k C_{ljih}) = g_{ik} B_{hjl} + g_{hj} B_{ikl} + g_{il} B_{hkj} + g_{hk} B_{ilj} + g_{ij} B_{hlk} + g_{hl} B_{ijk}. \tag{7}$$

The assertion follows now from (6) and (7).

THEOREM 1. *Let (M, g) be conformally symmetric with respect to the metric semi-symmetric connection* (1). *Then* $\dim M = 4$ *or*

$$p_j B_{kih} + p_h B_{kji} + p_i B_{khj} = 0.$$

Proof. From Proposition 1, by transvection with p^l, we obtain

$$p^r(\nabla_r C_{kjih} + \nabla_j C_{rkih} - \nabla_k C_{rjih}) = (1-n)(p_r p^r C_{kjih} + p_j B_{kih} - p_k B_{jih}),$$

whence, in view of (5) and Lemma 1,

$$(3-n)(p_r p^r C_{kjih} + p_j B_{kih} - p_k B_{jih}) = g_{ij} T_{hk} - g_{hj} T_{ik} - g_{ik} T_{hj} + g_{hk} T_{ij} + p_i B_{hjk} - p_h B_{ijk}, \tag{8}$$

where $T_{ij} = p^r B_{ijr} = T_{ji}$. Permuting in (8) the indices j, i, h cyclically, adding the resulting equations to (8) and applying Lemma 1, we get

$$(4-n)(p_j B_{kih} + p_h B_{kji} + p_i B_{khj}) = 0,$$

which, evidently, completes the proof.

THEOREM 2. *Let M be a Riemannian manifold admitting a metric semi-symmetric connection* (1) *such that*

$$\mathring{\nabla}_l \mathring{C}_{kjih} = \nabla_l C_{kjih}. \tag{9}$$

Then at each given point of M we have $p_j = 0$ or $C_{hijk} = 0$.

Proof. Differentiating $\mathring{C}_{kjih}$ covariantly and using (1) and Lemma 3, we get

$$\begin{aligned}\mathring{\nabla}_l \mathring{C}_{kjih} = \nabla_l C_{kjih} - (p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil}) \\ + g_{lk} B_{jih} + g_{li} B_{hkj} + g_{lj} B_{khi} + g_{lh} B_{ijk}.\end{aligned} \tag{10}$$

Equation (10), together with (9), implies

$$\begin{aligned}p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil} = g_{lk} B_{jih} + g_{li} B_{hkj} \\ + g_{lj} B_{khi} + g_{lh} B_{ijk}.\end{aligned} \tag{11}$$

Contracting (11) with g^{hl} and using Lemma 1, we get $B_{ijk} = 0$. Thus

$$p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil} = 0.$$

Setting $c_l = 0$ in Lemma 2 and applying the last equation, we easily obtain $p_j = 0$ or $C_{hijk} = 0$. This completes the proof.

COROLLARY 1. *If $\nabla_l C^h{}_{ijk} = a_l C^h{}_{ijk}$ and $\mathring{\nabla}_l \mathring{C}^h{}_{ijk} = a_l \mathring{C}^h{}_{ijk}$ and p_i does not identically vanish on M, then M is necessarily conformally flat.*

Proof. Let $x \in M$ be such that $C^h{}_{ijk}(x) = 0$. Then, since $\nabla_l C^h{}_{ijk} = a_l C^h{}_{ijk}$, C vanishes everywhere on M. The assertion follows now from Theorem 2.

PROPOSITION 2. *Let M be a Riemannian manifold admitting a metric semi-symmetric connection* (1) *such that*

$$\mathring{\nabla}_l \mathring{C}_{kjih} = \mathring{a}_l \mathring{C}_{kjih}, \qquad \nabla_l C_{kjih} = a_l C_{kjih}. \tag{12}$$

Then $p_r C^r{}_{ijk} = 0$ holds everywhere on M.

Proof. If M is conformally flat, then the assertion is trivial. Thus, in view of (12), we may assume that $C_{kjih} \neq 0$ at each point of M. Equation (10), together with (12) and Lemma 3, yields

$$\begin{aligned}p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil} - \psi_l C_{kjih} \\ = g_{lk} B_{jih} + g_{li} B_{hkj} + g_{lj} B_{khi} + g_{lh} B_{ijk},\end{aligned} \tag{13}$$

where $\psi_l = a_l - \mathring{a}_l$. This, by Lemma 1 and contraction with g^{kl}, gives $\psi_r C^r{}_{jih} = (1-n) B_{jih}$. Transvecting now (13) with $p^k p^h$, we obtain

$$(2p_l + \psi_l) T_{ji} = p_i T_{jl} + p_j T_{il} + p^r p_r (B_{ijl} + B_{jil}).$$

Now, with the help of the last result we can follow step by step a proof of Roter (see [4], p. 43) to obtain $B_{ijk} = 0$. This completes the proof.

THEOREM 3. *Let M be a Riemannian manifold which admits a metric semi-symmetric connection* (1) *such that conditions* (12) *hold. Then*

(a) $p_l C_{hijk} + p_k C_{hilj} + p_j C_{hikl} = 0$ *everywhere on M.*

(b) *If (M, g) is not conformally flat, then* $\mathring{a}_j = a_j - 2p_j$ *and* $p_r p^r = 0$.

Proof. Obviously, we can assume that $C \neq 0$ everywhere. Since $B_{ijk} = 0$, (13) yields

$$-\psi_l C_{kjih} + p_k C_{ljih} + p_j C_{klih} + p_i C_{kjlh} + p_h C_{kjil} = 0,$$

whence, using Lemma 2, we get $\psi_l = 2p_l$. Now, with the help of the last result, we can follow step by step a proof of Roter ([4], p. 44) to obtain our assertion.

THEOREM 4. *Let a Riemannian manifold admit a metric semi-symmetric connection* (1) *such that the function p satisfies equation* (a) *of Theorem* 3. *Then*

$$\mathring{\nabla}_l \mathring{C}_{kjih} = \nabla_l C_{kjih} - 2p_l C_{kjih}. \tag{14}$$

Proof. Assume (a) holds. Then, in view of Lemma 1, we get $B_{ijk} = 0$. Hence equation (10) can be written as

$$\mathring{\nabla}_l \mathring{C}_{kjih} = \nabla_l C_{kjih} - (p_k C_{ljih} + p_j C_{klih}) - (p_i C_{kjlh} + p_h C_{kjil}). \tag{15}$$

Substituting (a) into (15) and using Lemma 1, we obtain (14). This completes the proof.

COROLLARY 2. *Let M admit a metric semi-symmetric connection* (1) *such that p_i satisfies* (a). *Then M is conformally recurrent if and only if the condition* $\mathring{\nabla}_l \mathring{C}_{hijk} = \mathring{a}_l \mathring{C}_{hijk}$ *holds.*

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PEDAGOGICAL UNIVERSITY
OPOLE

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