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ON A LOWER BOUND OF THE SECOND EIGENVALUE OF THE LAPLACIAN ON AN EINSTEIN SPACE

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0. Introduction. Let (M, g) be a Riemannian manifold and let Δ be the Laplacian acting on C^∞-functions on (M, g). Then some results on lower bounds or upper bounds of the first eigenvalue of Δ are given in several situations (cf., e.g., [1]-[4] and [7]).

Let (M, g) be a compact orientable Einstein space with constant scalar curvature $S = m(m-1)K$, where $m = \dim M \geqslant 2$. Then the lower bound of the first eigenvalue λ_1 of Δ is mK. Furthermore, if (M, g) admits an eigenfunction f ($\Delta f = -mKf$) corresponding to mK, then (M, g) is isometric to a Euclidean m-sphere of constant curvature K (cf. Obata [6]).

In this paper* we prove the following theorem:

THEOREM. *Let (M, g) be a compact orientable Einstein space with constant positive scalar curvature $S = m(m-1)K$, where $m = \dim M \geqslant 2$. Let B be the minimum of the sectional curvature of (M, g) and assume that $B > 0$. Then*

(i) *there is no eigenvalue λ of Δ such that*

$$mK < \lambda \leqslant K + 2(m-1)B;$$

(ii) *if $B \neq K$, then the first eigenvalue λ_1 of Δ satisfies*

$$\lambda_1 > K + 2(m-1)B;$$

(iii) *generally, the second eigenvalue λ_2 of Δ satisfies*

$$\lambda_2 > K + 2(m-1)B.$$

It may be remarked that if (M, g) is a Euclidean m-sphere of constant curvature K, then the second eigenvalue λ_2 of Δ is $2(m+1)K = K + (2m+1)K$.

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1. Proof of the Theorem. Let (M, g) be a compact orientable Einstein space stated in the Theorem. Let f be a non-constant eigenfunction of Δ such that

$$\Delta f = -\lambda f, \tag{1}$$

where $\Delta f = g^{ij}\nabla_i\nabla_j f = \nabla^i\nabla_i f$, and ∇ denotes the Riemannian connection defined by $g = (g_{ij})$. By Obata's theorem in the introduction, we study only the case where $\lambda > mK$. We put

$$f_i = \nabla_i f \quad \text{and} \quad f_{ij} = \nabla_j\nabla_i f.$$

Now we define a $(0, 2)$-tensor field $A = (A_{ij})$ by

$$A_{ij} = f_{ij} + Kfg_{ij}. \tag{2}$$

Then we get $\nabla_k A_{ij} = \nabla_k f_{ij} + Kf_k g_{ij}$ and

$$\begin{aligned} \nabla_k A_{ij}\nabla^k A^{ij} &= \nabla_k f_{ij}\nabla^k f^{ij} + 2Kf^k\nabla_k\Delta f + K^2 m f_k f^k \\ &= \nabla_k f_{ij}\nabla^k f^{ij} + (mK^2 - 2K\lambda)f_k f^k, \end{aligned} \tag{3}$$

where

$$\nabla_k f_{ij}\nabla^k f^{ij} = \nabla_k(f_{ij}\nabla^k f^{ij}) - f_{ij}\nabla_k\nabla^k f^{ij}.$$

By applying the Ricci identity and using the Einstein condition, the second term of the right-hand side is calculated to be

$$-f_{ij}\nabla_k\nabla^k f^{ij} = -f^{ij}\nabla_j\nabla_i\Delta f + 2f^{ij}(R^h{}_{kj}{}^k f_{ih} + R^h{}_{ij}{}^k f_{hk}), \tag{4}$$

where $(R^i{}_{jkl})$ denotes the Riemannian curvature tensor field (cf. [8]).

At each point x of M, we take an orthonormal frame $\{E_i\}$ such that each E_i is an eigenvector of (f^i_j) at x and let σ_i be the corresponding eigenvalue. We estimate scalars with respect to this frame $\{E_i\}$. Then

$$\begin{aligned} -f^{ij}\nabla_j\nabla_i\Delta f &= \lambda f^{ij}f_{ij} = \lambda\sum_i(\sigma_i)^2 \\ &= \frac{\lambda}{m}\Big\{\sum_{i<j}(\sigma_i-\sigma_j)^2 + \Big(\sum_i\sigma_i\Big)^2\Big\} = \frac{\lambda}{m}\Big\{\sum_{i<j}(\sigma_i-\sigma_j)^2 + (\lambda f)^2\Big\}. \end{aligned} \tag{5}$$

On the other hand, we get (cf. [8])

$$2f^{ij}(R^h{}_{kj}{}^k f_{ih} + R^h{}_{ij}{}^k f_{hk}) = -2\sum_{i<j}K_{ij}(\sigma_i-\sigma_j)^2, \tag{6}$$

where K_{ij} denotes the sectional curvature for a 2-plane spanned by E_i and E_j.

Next we rewrite $f^{ij}f_{ij}$ and f_kf^k:

$$\begin{aligned} (7) \qquad f^{ij}f_{ij} &= \nabla_j(f_if^{ij}) - f^i\nabla_j\nabla_if_j \\ &= \nabla_j(f_if^{ij}) - f^i(\nabla_i\Delta f + (m-1)Kf_i) \\ &= \nabla_j(f_if^{ij}) + (\lambda - mK + K)f_kf^k, \end{aligned}$$

$$(8) \qquad f_kf^k = \nabla_k(ff^k) - f\Delta f = \nabla_k(ff^k) + \lambda f^2.$$

By (7) and (8) we get

$$\begin{aligned} (9) \qquad f^{ij}f_{ij} &= \nabla_j\big(f_if^{ij} + (\lambda - mK + K)ff^j\big) + (\lambda - mK + K)\lambda f^2 \\ &= \nabla_jN_1^j + (\lambda - mK + K)\lambda f^2, \end{aligned}$$

where N_1 (or N_2 and N_3, later) denotes some vector field on M the explicit expression of which we do not need. Eliminating $f^{ij}f_{ij}$ from (5) and (9), we obtain

$$(10) \qquad \lambda f^2 = \nabla_jN_2^j + \frac{1}{m(\lambda - mK + K) - \lambda}\sum_{i<j}(\sigma_i - \sigma_j)^2.$$

Applying (4)-(10), we can write (3) as

$$(11) \qquad \nabla_kA_{ij}\nabla^kA^{ij} = \nabla_jN_3^j + \sum_{i<j}\left\{\frac{\lambda - K}{m-1} - 2K_{ij}\right\}(\sigma_i - \sigma_j)^2.$$

Therefore, if $\lambda \leqslant K + 2(m-1)B$, then integrating (11) on M we obtain $\nabla_kA_{ij} = 0$. Since (M, g) is irreducible by $B > 0$, A_{ij} is of the form $A_{ij} = cg_{ij}$ for some constant c, and

$$f_{ij} + Kfg_{ij} = cg_{ij}.$$

Transvecting the last equation with g^{ij}, we get

$$\Delta f + mKf = mc.$$

Since f is an eigenfunction, integrating the last equation on M we get $c = 0$ and $\lambda = -mK$. This is a contradiction. Therefore, there is no eigenvalue λ such that $mK < \lambda \leqslant K + 2(m-1)B$. This proves (i) of the Theorem. (ii) follows from (i) and Obata's theorem. And, generally, we have (iii).

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