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NOTE ON METRIZATION OF QUASI-UNIFORM SPACES

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One of the classical metrization theorems is that if $(X, \mathscr{U})$ is a T_1 uniform space and $\mathscr{U}$ has a countable basis then $(X, \mathscr{U})$ is metrizable. If $(X, \mathscr{U})$ is a T_1 quasi-uniform space and $\mathscr{U}$ has a countable basis, then $(X, \mathscr{U})$ is quasi-metrizable; but $(X, \mathscr{U})$ is not necessarily even Hausdorff. In this paper we give necessary and sufficient conditions that a T_1 quasi-uniform space be metrizable.

Throughout, if $(X, \mathscr{U})$ is a quasi-uniform space and $x \in X$, then $\mathfrak{F}(x)$ will denote the filter generated by the filter-basis $\{U^{-1}[U[x]]: U \in \mathscr{U}\}$.

Standard results concerning quasi-uniform spaces which are assumed here may be found in [3].

Definition. A quasi-uniform space $(X, \mathscr{U})$ is *pre-metric* provided that for each $x \in X$, $\mathfrak{F}(x)$ is a convergent filter.

PROPOSITION 1. *Let $(X, \mathscr{U})$ be a complete quasi-uniform space. Then $(X, \mathscr{U})$ is pre-metric if and only if for each $V \in \mathscr{U}$ and each $x \in X$ there exists $W \in \mathscr{U}$ with $W \circ W \subset V$ and $p \in X$ such that $z \in V[p]$ whenever $W[x] \cap W[z] \neq \emptyset$.*

LEMMA ([1] or [2], p. 126). *Let N be the set of positive integers. A space X is metrizable if at every point $p \in X$ there exists a sequence $\beta(p) = \{W_n(p): n \in N\}$ of open neighborhoods such that:*

(a) *$W_{n+1}(p) \subset W_n(p)$ for each $n \geqslant 1$,*

(b) *$\bigcap\{W_n(p): n \in N\} = \{p\}$,*

(c) *$\beta(p)$ is a local basis at p,*

(d) *for each $p \in X$ and each $n \geqslant 1$, there is an $m \geqslant 1$ such that $W_m(q) \subset W_n(p)$ whenever $W_m(q) \cap W_m(p) \neq \emptyset$.*

THEOREM 1. *Let $(X, \mathscr{U})$ be a pre-metric quasi-uniform space whose associated topology $\mathscr{T}_{\mathscr{U}}$ is T_1. Then $(X, \mathscr{T}_{\mathscr{U}})$ is regular. Furthermore, if $\mathscr{U}$ has a countable basis, then $(X, \mathscr{T}_{\mathscr{U}})$ is metrizable.*

Proof. Since $\mathscr{T}_{\mathscr{U}}$ is T_1, a filter whose sets all contain a given point x cannot converge to any point other than x. Thus for each $x \in X$, $\mathfrak{F}(x)$ converges to x. Since the $\mathscr{T}_{\mathscr{U}}$-closure of $U[x]$ is contained in $U^{-1}[U[x]]$ it follows at once that $\mathscr{T}_{\mathscr{U}}$ is regular.

Now suppose that $\mathscr{U}$ has a countable basis. We define by induction a family of sets $\{W_n\colon n \in N\}$ which forms a base for $\mathscr{U}$ such that $W_{n+1} \subset W_n$ for each $n \geqslant 1$. For each $p \in X$ let $\beta(p) = \{\mathrm{int}[W_n[p]]\colon n \in N\}$. Conditions (a), (b) and (c) of the lemma are clearly satisfied.

(d) Let $p \in X$ and let n be a positive integer. Then there is a positive integer t such that $W_t \circ W_t \subset W_n$. Since $\mathfrak{F}(p)$ converges to p, there is a positive integer s such that $W_s \subset W_t$ and $W_s^{-1}[W_s[p]] \subset W_t[p]$. Suppose that $q \in X$ and that $\mathrm{int}[W_s[q]] \cap \mathrm{int}[W_s[p]] \neq \emptyset$. Then $q \in W_s^{-1}[W_s[p]] \subset W_t[p]$ so that $W_s[q] \subset W_s[W_t[p]] \subset W_t[W_t[p]] \subset W_n[p]$. It follows that $\mathrm{int}[W_s[q]] \subset \mathrm{int}[W_n[p]]$.

COROLLARY. *Let $(X, \mathscr{T})$ be a T_1 topological space. Then $(X, \mathscr{T})$ is metrizable if and only if there exists a compatible pre-metric quasi-uniformity which has a countable basis.*

It follows from Theorem 1 and [3], Theorem 3.17, iii, that a T_1 space $(X, \mathscr{T})$ is regular if and only if the Pervin quasi-uniformity for $(X, \mathscr{T})$ is pre-metric.

Example ([4], Example 2.7). Let X be the real line. Define $p(x, x) = 0$ for each $x \in X$. If $x \neq y$ define $p(x, y) = 1$ if x is rational and $p(x, y) = |x-y|/(1+|x-y|)$ if x is irrational. Then p is a quasi-metric. Let $\mathscr{U}$ be the quasi-uniformity generated by p in the natural way. Although $(X, \mathscr{U})$ is not pre-metric, $(X, \mathscr{U})$ is metrizable.

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