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ON THE COMPOSITION OF THE ARITHMETIC FUNCTIONS σ AND φ

BY

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1. Introduction. Let φ denote Euler's function and let σ denote the sum of the divisors function. In [2], Mąkowski and Schinzel consider the function $\sigma(\varphi(n))/n$, showing that

$$\liminf \frac{\sigma(\varphi(n))}{n} \leqslant \inf_{4|n} \frac{\sigma(\varphi(n))}{n} \leqslant \frac{1}{2}+\frac{1}{2^{34}-4}.$$

They ask if $\sigma(\varphi(n))/n \geqslant 1/2$ is true for all n, stating that Mrs. K. Kuhn has shown this inequality for all n with at most 6 prime factors. They also remark that even the weaker inequality

$$\inf \frac{\sigma(\varphi(n))}{n} > 0 \tag{1.1}$$

remains open.

In this note* we prove (1.1). The proof is elementary, the principal tool being Brun's method.

Throughout, the letter p denotes a prime.

2. The proof of (1.1). If T is a set of primes, let

$$s(T, x) = \sum_{x^{1/e} < p \leqslant x}' 1/p,$$

where $\sum'$ signifies that the primes p in the sum also have the property that $p-1$ is free of prime factors from T. We have the following result:

Lemma 2.1. *There is an absolute constant $c_1 \geqslant 1$ such that for any set of primes T and any x we have*

$$s(T, x) \leqslant c_1 \exp\Big(-\sum_{\substack{p \leqslant x \\ p \in T}} 1/p\Big).$$

Indeed, this follows from [1] (Theorem 2.2) and a partial summation.

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Note that if m is a natural number, then

$$\log\frac{\sigma(m)}{m} = \sum_{p^a \| m} \log(1+1/p+\ \dots\ +1/p^a) = \sum_{p|m} 1/p + O(1),$$

$$\log\frac{\varphi(m)}{m} = \sum_{p|m} \log(1-1/p) = -\sum_{p|m} 1/p + O(1).$$

Given a natural number n, let $S(n)$ denote the set of prime factors of $\varphi(n)$. Thus

$$\log\frac{\sigma(\varphi(n))}{n} = \log\frac{\sigma(\varphi(n))}{\varphi(n)} + \log\frac{\varphi(n)}{n} = \sum_{p\in S(n)} 1/p - \sum_{p|n} 1/p + O(1). \tag{2.1}$$

Let $S'(n)$ denote the set of primes p such that every prime in $p-1$ lies in $S(n)$. Thus from (2.1) we have

$$\log\frac{\sigma(\varphi(n))}{n} \geqslant \sum_{p\in S(n)} 1/p - \sum_{p\in S'(n)} 1/p + O(1).$$

We conclude that (1.1) will follow from the following theorem:

THEOREM 2.2. *There is an absolute constant c_2 such that*

$$\sum_{p\in S'} 1/p - \sum_{p\in S} 1/p \leqslant c_2$$

for any finite set of primes S, where S' denotes the set of primes p such that every prime in $p-1$ lies in S.

Proof. Let S be an arbitrary finite set of primes, let k be a natural number such that $S \subset [1, \exp(e^k)]$, and let T be the (infinite) set of primes that do not belong to S. For each natural number i, let

$$\beta_i = \sum_{\substack{p\in S' \\ \exp(e^{i-1}) < p \leqslant \exp(e^i)}} 1/p,$$

so that

$$\sum_{p\in S'} 1/p = \tfrac{1}{2} + \sum_{i=1}^{\infty} \beta_i. \tag{2.2}$$

Note that from Lemma 2.1 we have

$$\beta_i = s(T, \exp(e^i)) \leqslant c_1 \exp\Big(-\sum_{\substack{p\in T \\ p\leqslant \exp(e^i)}} 1/p\Big) \tag{2.3}$$

for each natural number i. Let

$$b_i = \sum_{\substack{p\in T \\ \exp(e^{i-1}) < p \leqslant \exp(e^i)}} 1/p \quad \text{for } i = 1, 2, \dots$$

Note that we may assume $2 \notin T$, for otherwise $S' = \{2\}$ and the theorem is trivially true. Thus

$$\sum_{\substack{p \in T \\ p \leqslant \exp(e^i)}} 1/p = \sum_{j=1}^{i} b_j \quad \text{for } i = 1, 2, \ldots \tag{2.4}$$

Since

$$b_i' := \sum_{\exp(e^{i-1}) < p \leqslant \exp(e^i)} 1/p = 1 + O(e^{-i}), \quad i = 1, 2, \ldots, \tag{2.5}$$

we infer from (2.3) and (2.4) that

$$\beta_i \leqslant \min(1, c_1 \exp(-(b_1 + \ldots + b_i))) + O(e^{-i})$$

for every natural number i. We conclude from (2.2) that

$$\sum_{p \in S'} 1/p - \sum_{p \in S} 1/p = \tfrac{1}{2} + \sum_{i=1}^{\infty} \beta_i - \sum_{p \leqslant \exp(e^k)} 1/p + \sum_{\substack{p \in T \\ p \leqslant \exp(e^k)}} 1/p \tag{2.6}$$

$$\leqslant \sum_{i=1}^{\infty} \min(1, c_1 \exp(-(b_1 + \ldots + b_i))) - k + b_1 + \ldots + b_k + O(1).$$

Now, for $i > k$, we infer from (2.5) that $b_i = b_i' = 1 + O(e^{-i})$, so that

$$\sum_{i > k} \min(1, c_1 \exp(-(b_1 + \ldots + b_i))) \ll \sum_{i > k} e^{-(i-k)} \ll 1. \tag{2.7}$$

Thus from (2.5)–(2.7) we will have Theorem 2.2 once we prove the following lemma:

LEMMA 2.3. *If $c \geqslant 1$ is fixed and if*

$$f(b_1, \ldots, b_k) := \sum_{i=1}^{k} \min(1, c\exp(-(b_1 + \ldots + b_i))) - k + b_1 + \ldots + b_k,$$

then the maximum value of f on $[0, 1]^k$ is less than $e/(e-1) + \log c$ for any natural number k.

Note that the letters $b_1, \ldots, b_k$ appearing in (2.6) satisfy $0 \leqslant b_i \leqslant b_i'$ for each i, rather than $0 \leqslant b_i \leqslant 1$. However, the maximum value of f on $[0, b_1'] \times \ldots \times [0, b_k']$ is less than or equal to the maximum value of f on

$$[0, \max(1, b_1')] \times \ldots \times [0, \max(1, b_k')],$$

which, by (2.5), is within $O(1)$ of the maximum value of f on $[0, 1]^k$. Thus Lemma 2.3 suffices for the completion of the proof of Theorem 2.2 and (1.1).

Proof of Lemma 2.3. Let k be given and assume the maximum value of f is attained at $(B_1, \ldots, B_k) \in [0, 1]^k$. First note we may assume that

$$B_1 \leqslant B_2 \leqslant \ldots \leqslant B_k,$$

for if $B_{i+1} < B_i$, then interchanging these two numbers will not make f smaller. If

$$c\exp\left(-(B_1 + \dots + B_k)\right) \geqslant 1,$$

then

$$f(B_1,\dots,B_k) = B_1 + \dots + B_k \leqslant \log c.$$

So we may assume there is a first subscript i_0 such that

$$c\exp\left(-(B_1 + \dots + B_{i_0})\right) < 1.$$

Then $B_1 + \dots + B_{i_0-1} \leqslant \log c$, so that

$$\text{(2.8)}\qquad f(B_1,\dots,B_k) \leqslant i_0 + \sum_{i=i_0+1}^{k} \exp\left(-(B_{i_0+1} + \dots + B_i)\right) - k + B_1 + \dots + B_k$$

$$\leqslant \sum_{i=i_0+1}^{k} \exp\left(-(B_{i_0+1} + \dots + B_i)\right) - (k-i_0) + B_{i_0+1} + \dots + B_k + 1 + \log c.$$

Let

$$g(b_{i_0+1},\dots,b_k) = \sum_{i=i_0+1}^{k} \exp\left(-(b_{i_0+1} + \dots + b_i)\right) - (k-i_0) + b_{i_0+1} + \dots + b_k$$

and say the maximum of g on $[0, 1]^{k-i_0}$ is assumed at $(A_{i_0+1},\dots,A_k)$. Thus from (2.8) we have

$$\text{(2.9)}\qquad f(B_1,\dots,B_k) \leqslant g(A_{i_0+1},\dots,A_k) + 1 + \log c.$$

As above, we may assume $A_{i_0+1} \leqslant \dots \leqslant A_k$. If $A_{i_0+1} = \dots = A_k = 1$, then from (2.9) we have

$$f(B_1,\dots,B_k) \leqslant \sum_{j=1}^{k-i_0} e^{-j} + 1 + \log c < \frac{e}{e-1} + \log c.$$

Thus we may assume there is a greatest index i_1 with $A_{i_1} < 1$.

If $A_{i_1} = 0$, then from (2.9) we have

$$f(B_1,\dots,B_k) \leqslant i_1 - i_0 + \sum_{j=1}^{k-i_1} e^{-j} - (k-i_0) + (k-i_1) + 1 + \log c$$

$$< \frac{e}{e-1} + \log c.$$

Thus we may assume $0 < A_{i_1} < 1$. Then

$$\frac{\partial g}{\partial b_{i_1}}(A_{i_0+1},\dots,A_k) = 0,$$

so that

$$(2.10)\quad 1 = \sum_{i=i_1}^{k} \exp\left(-(A_{i_0+1}+\ldots+A_i)\right)$$
$$= \left[\exp\left(-(A_{i_0+1}+\ldots+A_{i_1})\right)\right]\left(1+\exp(-1)+\ldots+\exp\left(-(k-i_1)\right)\right).$$

This implies

$$A_{i_0+1}+\ldots+A_{i_1} < \log\frac{e}{e-1}.$$

Thus from (2.9) and (2.10) we have

$$f(B_1,\ldots,B_k) < (i_1-i_0-1)+1-(k-i_0)+\log\frac{e}{e-1}+(k-i_1)+1+\log c$$
$$= \log\frac{e}{e-1}+1+\log c < \frac{e}{e-1}+\log c,$$

which proves the lemma.

REFERENCES

[1] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press, London 1974.
[2] A. Mąkowski and A. Schinzel, *On the functions $\varphi(n)$ and $\sigma(n)$*, Colloq. Math. 13 (1964), pp. 95–99.

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