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# ON LOCALLY HOMEOMORPHIC IMAGES OF IRREDUCIBLE CONTINUA

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In [4] Lelek raises the following question:

Suppose that $X$ is an irreducible continuum and that $f$ is a local homeomorphism which maps $X$ onto the continuum $Y$. Does it follow that $Y$ is irreducible?

In this note [1] we answer two special cases of this question. First, we show that if $X$ is an indecomposable continuum, then $Y$ is also indecomposable (and hence irreducible); and second, that if $X$ is an irreducible continuum of type $\lambda$, then $Y$ is also such a continuum.

All topological spaces considered below will be assumed to be Hausdorff. In addition the words "map" and "mapping" will always refer to continuous functions.

Definition 1. A topological space is said to be a *continuum* if it is a compact, connected, metric space.

Definition 2. A continuum $X$ is said to be *irreducible between two points $a$ and $b$ in $X$* if no proper subcontinuum of $X$ contains both $a$ and $b$. A continuum which is irreducible between some two of its points is said to be *irreducible*.

Definition 3. A function $f$ from a topological space $X$ to a topological space $Y$ is said to be a *local homeomorphism* if for every $x \in X$ there exists a neighborhood $U_x$ of $x$ such that $f(U_x)$ is a neighborhood of $f(x)$ and such that $f$ restricted to $U_x$ is a homeomorphism between $U_x$ and $f(U_x)$.

LEMMA 1. *If $X$ is a compact topological space and $f$ is a local homeomorphism which maps $X$ onto the space $Y$, then there exists an integer $n$ such that, for every $y \in Y$, $f^{-1}(y)$ contains at most $n$ points.*

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[1] This research was conducted while the author was participating in an exchange of scientists administrated jointly by the Polish Academy of Sciences and the National Academy of Sciences (U.S.A.).

Proof. Let $X, f$ and $Y$ be as above. For every $x \in X$ let $U_x$ be an open neighborhood of $x$ such that $f$ restricted to $U_x$ is one-to-one. Since $X$ is compact, some finite number of these sets, say $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$, cover $X$. For any $y \in Y$ it is evident that $f^{-1}(y)$ can meet $U_{x_i}$, $i = 1, 2, \ldots, n$, in at most one point.

Remark 1. If a mapping $f$ is a local homeomorphism, then it is an open mapping.

Definition 4. A mapping $f$ from a space $X$ onto a space $Y$ is said to be *confluent* if, given any subcontinuum $K$ of $Y$ and any component $C$ of $f^{-1}(K)$, it follows that $f(K) = C$ (2).

The following is a corollary of a theorem proved by Whyburn in [5] (theorem 7.5, p. 148):

THEOREM 1. *If $f$ is an open mapping from a continuum $X$ onto a continuum $Y$, then $f$ is confluent.*

COROLLARY 1. *If $f$ is a local homeomorphism which maps a continuum $X$ onto a continuum $Y$, then $f$ is confluent.*

COROLLARY 2. *If $f$ is a local homeomorphism which maps the continuum $X$ onto the continuum $Y$, and $C$ is any subcontinuum of $Y$, then $f^{-1}(C)$ contains finitely many components.*

Proof. This follows immediately from lemma 1 and corollary 1.

Definition 5. A continuum $X$ is said to be *indecomposable* if there do not exist proper subcontinua $P$ and $Q$ of $X$ such that $P \cup Q = X$.

The following two theorems are well known (see e.g. [3], theorem 2, p. 145, and theorem 7′, p. 150):

THEOREM 2. *Every indecomposable continuum is irreducible.*

THEOREM 3. *A continuum $X$ is indecomposable if and only if every proper subcontinuum of $X$ has void interior.*

THEOREM 4. *If $X$ is an indecomposable continuum and $f$ is a local homeomorphism which maps $X$ onto the space $Y$, then $Y$ is an indecomposable continuum (and hence $Y$ is irreducible).*

Proof. Let $X, f$ and $Y$ be as above and suppose that $Y$ fails to be indecomposable (certainly $Y$ is a continuum). Then, by theorem 3, $Y$ contains a proper subcontinuum $C$ with non-void interior. But, by corollary 2, $f^{-1}(C)$ is a finite union of continua. Thus one of these continua must have non-void interior. This contradicts theorem 3.

Definition 6. An irreducible continuum $X$ is said to be *of type $\lambda$* if there is a monotone mapping $g$ from $X$ onto the unit interval $I$ such

(2) The notion of confluence was introduced by Charatonik in [1].

that, for every $t \in I$, $g^{-1}(t)$ has void interior in $X$. The sets $g^{-1}(t)$ for each $t \in I$ are called *tranches* of $X$ [3].

Remark 2. Suppose that $X$ is an irreducible continuum of type $\lambda$ and that $g$ is as above. If $C$ is a subcontinuum of $X$ and $t_1$ and $t_2$ are points of $I$ such that $t_1 < t_2$ and $C \cap g^{-1}(t_1) \neq \emptyset \neq C \cap g^{-1}(t_2)$, then $g^{-1}(t) \subset C$ for every $t \in I$ such that $t_1 < t < t_2$.

LEMMA 2. *If $X$ is an irreducible continuum of type $\lambda$ and $C$ is a subcontinuum of $X$ with void interior, then $C$ is contained in some tranche of $X$.*

Proof. This follows immediately from remark 2.

We now make the standing assumption that $X$ is an irreducible continuum of type $\lambda$, that $g_1$ is as in definition 6 and that $f$ is a local homeomorphism which maps $X$ onto the space $Y$. Note that $Y$ is a continuum.

LEMMA 3. *If $T$ is a tranche of $X$, then $f(T)$ has void interior in $Y$.*

Proof. Suppose that $X$ and $T$ are as above and that $f(T)$ contains an open subset $U$ of $Y$. Then for each $x \in f^{-1}(U) \cap T$ let $U_x$ be a neighborhood of $x$ such that

(i) $f(\mathrm{Cl}(U_x)) \subset U$,

(ii) $f(U_x)$ is open in $Y$,

(iii) $f$ restricted to $U_x$ is a homeomorphism onto $f(U_x)$,

(iv) $f$ restricted to $\mathrm{Cl}(U_x)$ is one-to-one.

Since $f^{-1}(U) \cap T$ is Lindelöf, some countable subcollection of the $U_x$'s, say $U_{x_1}, U_{x_2}, \ldots$, covers $f^{-1}(U) \cap T$. Moreover, since $f(T) \supset U$, we have

$$U = \bigcup \{f(U_{x_i} \cap T)\colon i = 1, 2, \ldots\}.$$

This implies that

$$(*) \qquad U = \bigcup \{f(\mathrm{Cl}(U_{x_i} \cap T))\colon i = 1, 2, \ldots\}.$$

Note that since each of the sets $\mathrm{Cl}(U_{x_i} \cap T)$ is compact, each of the sets $f(\mathrm{Cl}(U_{x_i} \cap T))$ is closed in $Y$. Finally, note that each of the sets $f(\mathrm{Cl}(U_{x_i} \cap T))$ has void interior in $U$ (since $T$ has void interior in $X$, conditions (i)-(iv) imply that $f(\mathrm{Cl}(U_{x_i} \cap T))$ has void interior in $f(\mathrm{Cl}(U_{x_i})) \subset U$). Equation (*) then implies that $U$ is the countable union of closed sets with void interior in $U$, violating the Baire category theorem.

LEMMA 4. *If $T_1$ and $T_2$ are tranches of $X$ and $f(T_1) \cap f(T_2') \neq \emptyset$, then $f(T_1) = f(T_2)$.*

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[3] Kuratowski ([3], theorem 3, p. 153) has given the following characterization of irreducible continua of type $\lambda$: An irreducible continuum $X$ is of type $\lambda$ if and only if every indecomposable subcontinuum of $X$ has void interior.

Proof. Let $T_1$ and $T_2$ be as above. Consider the set $f^{-1}(f(T_1))$. By corollary 2, this set has finitely many components. Moreover, since $f$ is an open map (remark 1), $f$ is confluent (corollary 1) and $f(T_1)$ has void interior (lemma 3), it follows that each component of $f^{-1}(f(T_1))$ has void interior in $X$. Lemma 2 then implies that each component of $f^{-1}(f(T_1))$ is contained in some tranche of $X$. Since $f(T_1) \cap f(T_2) \neq \emptyset$, some component of $f^{-1}(f(T_1))$ must then be contained in $T_2$. The confluence of $f$ now implies that $f(T_1) \subset f(T_2)$. A similar argument shows that $f(T_2) \subset f(T_1)$.

We now define an equivalence relation $=_f$ on $Y$ by $x =_f y$ if and only if $x, y \in f(T)$ for some tranche $T$ of $X$ (lemma 4 implies that $=_f$ is a genuine equivalence relation). The proof of the following lemma is perfectly straight-forward and is left to the reader:

LEMMA 5. *$=_f$ is an upper semi-continuous equivalence relation on $Y$.*

Lemma 5 implies that the quotient space $Y/=_f$ is a continuum. Let $\sim$ be the equivalence relation of $X$ generated by the map $g_1$. Then $X/\sim\, = I$, the unit interval. Moreover, $f$ induces a function $f^*: I \to Y/=_f$ given by $f^*(g_1(T)) = f(T)$ for every tranche $T$ of $X$ (recall that points of $Y/=_f$ are equivalence classes of the form $f(T)$, where $T$ is tranche of $X$). The proof of the next lemma is also straight-forward and is once again left to the reader.

LEMMA 6. *$f$ is an open mapping. Thus, in particular, $f$ is a confluent mapping.*

In [2] (corollary 20, p. 32) Charatonik proves the following theorem:

THEOREM 5. *If $f$ is a confluent mapping whose domain is the unit interval $I$, then $f(I)$ is an arc.*

COROLLARY 3. *$Y/=_f$ is an arc.*

Thus $Y$ admits an upper semi-continuous decomposition whose hyper-space is the unit interval. Let $g_2$ be the map from $Y$ onto $I$ induced by $=_f$. The definition of $=_f$ and lemma 3 imply that $g_2$ is monotone and that, for every $t \in I$, $g_2^{-1}(t)$ has void interior in $Y$. Thus, in order to show that $Y$ is an irreducible continuum of type $\lambda$, it remains only to show that $Y$ is irreducible.

LEMMA 7. *If $a$ and $b$ are two points of $Y$ such that $a \in g_2^{-1}(0)$ and $b \in g_2^{-1}(1)$, then $Y$ is irreducible between $a$ and $b$.*

Proof. Let $a$ and $b$ be as above and let $C$ be a subcontinuum of $Y$ which is irreducible between $a$ and $b$. Evidently, $C$ meets $g_2^{-1}(t)$ for every $t \in I$. This implies that $f^{-1}(C)$ meets every tranche of $X$. Corollary 2 implies that $f^{-1}(C)$ has finitely many components. Using remark 2 and the fact that every tranche of $X$ has void interior in $X$, it is now easy to show that $f^{-1}(C) = X$. Therefore $C = Y$.

The following theorem is now clear:

THEOREM 6. *If $X$ is an irreducible continuum of type $\lambda$ and $f$ is a local homeomorphism which maps $X$ onto the space $Y$, then $Y$ is an irreducible continuum of type $\lambda$.*

It should be noted that neither theorem 4 nor theorem 6 remains true under the relaxed hypothesis that $f$ is an open mapping. In [1] (p. 218) Charatonik gives an example of an indecomposable continuum (van Danzig's solenoid) which admits an open mapping onto a circle. Also in [1] (p. 216) Charatonik gives an example of an irreducible continuum of type $\lambda$ which admits an open mapping onto a circle.

### REFERENCES

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