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BOHR LOCAL PROPERTIES OF $C_A(T)$

BY

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Let Z be the group of relative integers, T its dual group, and Λ a subset of Z. We denote by $C_\Lambda(T)$ the closed subspace of $C(T)$ (continuous functions on T) which is spanned by $e^{i\lambda t}$ ($\lambda \in \Lambda$).

We consider the following problem:

(P) is a *Bohr local property* of $C_\Lambda(T)$ if for every $k \in Z$ there exists a neighborhood $v(k)$ in the Bohr compactification of Z such that either $\Lambda \cap v(k)$ is empty or $C_{\Lambda \cap v(k)}(T)$ has property (P). Does it imply that $C_\Lambda(T)$ has property (P)?

We give (Theorem 2) an example of a set Λ (a subset of prime numbers) such that $C_\Lambda(T)$ has a closed subspace isomorphic to c_0 and for every $k \in Z$ there exists $v(k)$ such that $C_{\Lambda \cap v(k)}(T)$ is either empty or one-dimensional. This gives a negative answer to the problem for some properties (P) (Theorem 3). We will study more generally sets of first kind (Definition 2 below).

The same kind of problems was studied before by Y. Meyer and G. Godefroy for closed subspaces $L^1_\Lambda(T)$ of $L^1(T)$. ($L^1_\Lambda(T)$ is spanned by $e^{i\lambda t}$ ($\lambda \in \Lambda$).) A set $\Lambda \subset Z$ is called a *Riesz set* if $L^1_\Lambda(T) = M_\Lambda(T)$, where $M_\Lambda(T)$ is the space of Radon measures μ on T such that $\hat{\mu}(k) = 0$ for every $k \in Z \backslash \Lambda$. By [8], the property *$\Lambda$ is a Riesz set* is a local property of $L^1_\Lambda(T)$. Some other local properties of $L^1_\Lambda(T)$ are studied in [3], Theorem 2.3.

A set $\Lambda \subset Z$ is called a *Rosenthal set* if $C_\Lambda(T) = L^\infty_\Lambda(T)$. One aim of this paper is to answer the following question of G. Godefroy: is the property *Λ is a Rosenthal set* a local one for $C_\Lambda(T)$?

Notation and definitions are in the first section. Section 2 contains the definition of the announced set Λ and the properties (P) for which it is a counterexample. In Section 3 we give some sufficient conditions on a set $\Lambda \subset Z$ for $C_\Lambda(T)$ to have a closed subspace isomorphic to c_0. They are more general than the ones used in Section 2. We apply them to sets $\Lambda \cap v(k)$, where Λ is the set of prime numbers or $\Lambda = \{j^s\}_{j \geq 1}$ ($s \geq 1$ a fixed integer) and $k \in Z$ lies in the closure of Λ in the Bohr compactification of Z.

We thank S. Hartman for having brought [4] to our attention and for very fruitful discussions on this paper.

1. Notation and definitions. We will only consider sets $\Lambda \subset Z$ such that $\Lambda = (\lambda_j)_{j \geq 1}$, where $(\lambda_j)_{j \geq 1}$ is an increasing sequence. It is easy to extend definitions and results to the case where $(\lambda_j)_{j \in Z}$ is increasing.

We denote by T the group $R/2\pi Z$ provided with its usual topology, and by T_d the same group provided with the discrete topology. We denote by $\bar{Z}$ the Bohr compactification of Z, i.e., the dual group of T_d.

Q denotes the group of rational numbers in T. The spaces $C(T)$ and $M(T)$ were defined in the introduction. $L^1(T)$ is the space of classes of integrable functions on T with respect to Haar measure. $L^\infty(T)$ denotes its dual space, and $C''(T)$ denotes the bidual of $C(T)$. Let Λ be a subset of Z. $C_\Lambda(T)$, $L^1_\Lambda(T)$ and $M_\Lambda(T)$ were already defined. $L^\infty_\Lambda(T)$ is the dual of $L^1(T)/L^1_{Z\setminus\Lambda}(T)$ and $C_\Lambda(T)$ is a closed subspace of $L^\infty_\Lambda(T)$. $\bar{\Lambda}$ is the closure of Λ in $\bar{Z}$.

Let D be a discrete topological space. $c_0(D)$ is the completion of finitely supported functions for the norm

$$\|f\| = \sup_{t\in D} |f(t)|.$$

The dual of $c_0(D)$ is denoted by $l^1(D)$, the bidual is denoted by $l^\infty(D)$. For $D = N$ we write only c_0, l^1, l^∞.

DEFINITION 1. A real sequence $(u_j)_{j\geqslant 1}$ is *uniformly distributed modulo* 1 if

$$\forall k \in Z\setminus\{0\} \;\; n^{-1} \sum_{j=1}^{n} \exp(2i\pi k u_j) \to 0 \qquad (n \to +\infty).$$

DEFINITION 2 ([4]). Let $\Lambda = (\lambda_j)_{j\geqslant 1} \subset Z$ and let D be the set of t's $\in T$ for which $(\lambda_j t)_{j\geqslant 1}$ is not uniformly distributed modulo 1. We will say that Λ is a *set of first kind* if D is countable, and a *set of first kind* (Q) if D is a subset of Q (which will be the case in the examples).

The following sequence of functions is associated to every $\Lambda = (\lambda_j)_{j\geqslant 1}$:

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi\lambda_j t).$$

It is classically used in harmonic analysis (see, e.g., [4] and [7]). It is a uniformly bounded sequence in $C_\Lambda(T)$ which converges to 0 at every $t \in T\setminus D$ and to 1 at $t = 0$. If Λ is a set of first kind, a subsequence $(f_{n_k})_{k\geqslant 1}$ converges pointwise on T to a function l which is supported by D and such that $l(0) = 1$. In particular, $(f_{n_k})_{k\geqslant 1}$ is a weak Cauchy sequence which does not weakly converge.

DEFINITION 3 ([4]). Let $\Lambda = (\lambda_j)_{j\geqslant 1} \subset Z$ and $\Lambda' \subset \Lambda$. For $n \geqslant 1$ let

$$\delta(n) = \operatorname{Card}\{j \mid 1 \leqslant j \leqslant n,\; \lambda_j \in \Lambda'\}.$$

We say that Λ' has an *upper positive density with respect to* Λ if

$$\overline{\lim_n}\, n^{-1}\delta(n) > 0.$$

If $n^{-1}\delta(n)$ has a positive limit $(n \to +\infty)$, we say that Λ' has a *positive density with respect to* Λ.

2. We first give a sufficient condition for $C_\Lambda(T)$ to have a closed subspace isomorphic to c_0.

THEOREM 1. *Let $\Lambda = (\lambda_j)_{j\geq 1} \subset Z$ be a set of first kind and let R be an arithmetic progression such that $\Lambda \cap R$ has an upper positive density with respect to Λ. Let*

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi\lambda_j t).$$

Then

(a) *$\Lambda \cap R$ is a set of first kind.*

(b) *Let us assume that a subsequence of $(f_n)_{n\geq 1}$ converges pointwise to $l \in c_0(T_d)$. Then $C_\Lambda(T)$ has a closed subspace isomorphic to c_0. $C_{\Lambda\cap R}(T)$ also has one if either $\Lambda \cap R$ has a positive density with respect to Λ or if $(f_n)_{n\geq 1}$ converges pointwise to $l \in c_0(T_d)$.*

Proof. (a) Let $R = j_0 + qZ$ $(j_0 \in Z,\ q \in N\backslash 0)$. Let μ be the measure on T defined by

$$\mu = \exp(2i\pi j_0 t) q^{-1} \sum_{p=0}^{q-1} \delta_{pq^{-1}}.$$

Then $\hat{\mu}(j) = 1$ if $j \in R$ and $\hat{\mu}(j) = 0$ if $j \in Z\backslash R$. Let

$$f_n'(t) = \delta^{-1}(n) \sum_{1\leq j\leq n, \lambda_j \in \Lambda\cap R} \exp(2i\pi\lambda_j t).$$

Then

$$f_n' = n\delta^{-1}(n) f_n * \mu.$$

By hypothesis, $(f_n)_{n\geq 1}$ converges pointwise to 0 outside a countable set $D \subset T$; hence $(f_n')_{n\geq 1}$ converges pointwise to 0 outside the countable set

$$D' = \bigcup_{0\leq p\leq q-1} (D + pq^{-1}).$$

(b) Let $\mu \in M(T)$ and let $\mu = \mu_1 + \mu_2$, where μ_2 is the atomic part of μ. If $(f_{n_k})_{k\geq 1}$ converges pointwise to $l \in c_0(T_d)$, then

$$\langle f_{n_k}, \mu\rangle = \langle f_{n_k}, \mu_1 + \mu_2\rangle \to \langle l, \mu_2\rangle = \langle l, \mu\rangle;$$

hence l defines an element of $C_\Lambda^{\perp\perp}(T) \subset C''(T)$.

By Proposition 3.1 of [5] or Lemma 4 of [6], $C_\Lambda(T)$ has a closed subspace isomorphic to c_0.

If $n^{-1}\delta(n) \to \delta$ $(\delta > 0)$, the subsequence $(f_{n_k}')_{k\geq 1}$ converges pointwise to $\delta l * \mu$ which is again a non-zero function in $c_0(T_d)$. The same is true if $(f_n)_{n\geq 1}$ converges pointwise to $l \in c_0(T_d)$ and $n_k^{-1}\delta(n_k) \to \delta$ $(\delta > 0)$. Thus $\Lambda \cap R$ satisfies the same hypothesis as Λ and the same conclusion holds.

EXAMPLES. 1. Let $P = (p_j)_{j \geq 1}$ be the sequence of prime numbers. By Vinogradov's theorem ([2], Theorem 9.4), P is a set of first kind $(\boldsymbol{Q})$. For every $t \in \boldsymbol{Q}$ the sequence

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi p_j t)$$

converges to $l(t)$ and $l \in c_0(\boldsymbol{Q})$ ([2], p. 349, and the proof of Theorem 9.4).

Let $R = j_0 + q\boldsymbol{Z}$ ($j_0, q \in \boldsymbol{N}\backslash 0$). If j_0 and q are not relatively coprime, $P \cap R$ is either empty or contains one point. If j_0 and q are relatively coprime, $P \cap R$ has a positive density with respect to P ([2], Theorem 2.5 and p. 280).

So P and $P \cap R$ (if $P \cap R$ contains more than one point) satisfy the hypothesis, and hence the conclusion of Theorem 1. The fact that $P \cap R$ is a set of first kind $(\boldsymbol{Q})$ was already mentioned in the proof of Theorem 4 in [4].

2. Let s be an integer ≥ 1. Let $\Lambda = (j^s)_{j \geq 1}$ and $R = j_0 + q\boldsymbol{Z}$ ($j_0 \in \boldsymbol{Z}$, $q \in \boldsymbol{N}\backslash 0$). If $\Lambda \cap R$ contains j, it contains obviously $(j + q\boldsymbol{N})^s$; hence $\Lambda \cap R$ has a positive upper density with respect to Λ.

By Satz 9 of [11], $\Lambda = (j^s)_{j \geq 1}$ is a set of first kind $(\boldsymbol{Q})$. Let $t = a/q \in \boldsymbol{Q}$ $(0 \leq a < q)$. Then

$$f_n\left(\frac{a}{q}\right) = n^{-1} \sum_{j=1}^{n} \exp\left(2i\pi j^s \frac{a}{q}\right) = n^{-1} \sum_{l=1}^{q} \exp\left(2i\pi l^s \frac{a}{q}\right) A_{n,l,q},$$

where $A_{n,l,q} = \operatorname{Card}\{j \mid 1 \leq j \leq n,\ j \equiv l(q)\}$. As

$$n/q - 1 \leq A_{n,l,q} \leq n/q,$$

we have

$$f_n(a/q) \to f_q(a/q) \qquad (n \to +\infty).$$

By Lemma 2.4 of [10], for every $\varepsilon > 0$ there exists a constant $C(s, \varepsilon)$ such that

$$|f_q(a/q)| \leq C(s, \varepsilon) q^{\varepsilon - 1/2^{s-1}}, \qquad s \geq 2;$$

hence the function $a/q \rightsquigarrow f_q(a/q)$ lies in $c_0(\boldsymbol{Q})$. If $s = 1$, we have $f_q(0) = 1$ and $f_q(a/q) = 0$ if $a \neq 0$.

So $\Lambda = \{j^s\}_{j \geq 1}$ and $\Lambda \cap R$ (as soon as it is not empty) satisfy the hypothesis, and hence the conclusion of Theorem 1.

We will need one more property of the set P.

LEMMA 1 ([8]). *Let P be the set of prime numbers. For every $k \in \boldsymbol{Z}\backslash\{1, -1\}$ there exists a neighborhood $v(k)$ in $\bar{\boldsymbol{Z}}$ such that $v(k) \cap P$ is either empty or reduced to $\{k\}$.*

Let us recall the proof: for every $q \in \boldsymbol{Z}\backslash 0$, $\overline{q\boldsymbol{Z}}$ is an open set in $\bar{\boldsymbol{Z}}$ and $\overline{q\boldsymbol{Z}} \cap \boldsymbol{Z} = q\boldsymbol{Z}$. Take $v(0) = \overline{4\boldsymbol{Z}}$ and $v(k) = k + \overline{3k\boldsymbol{Z}}$ if $k \notin \{0, 1, -1\}$.

Here is the announced example:

THEOREM 2. *Let P be the set of prime numbers, $R = 2+5Z$ and $\Lambda = P \cap R$. Then:*

(a) *For every $k \in Z$ there exists a neighborhood $v(k)$ in $\bar{Z}$ such that $\Lambda \cap v(k)$ is either empty or reduced to $\{k\}$.*

(b) *$C_\Lambda(T)$ has a closed subspace isomorphic to c_0.*

Proof. (a) For $k \neq \pm 1$, $v(k)$ is chosen as in Lemma 1. $v(1) = 1+\overline{5Z}$ and $v(-1) = -1+\overline{5Z}$ are disjoint from R.

(b) follows from the recalled properties of P and Theorem 1.

Let us now look for which properties (P) this example shows that (P) is not a Bohr local property of $C_\Lambda(T)$.

A Banach space X has the *Schur property* if every weak Cauchy sequence in X is norm convergent. Every finite dimensional space as well as l^1 have the Schur property but c_0 does not. A Banach space X has the *Radon–Nikodým property* if every bounded linear operator: $L^1(T) \to X$ is representable by a strongly measurable function: $T \to X$. By [5], $C_\Lambda(T)$ has this property iff Λ is a Rosenthal set (and $L^1_\Lambda(T)$ has this property iff Λ is a Riesz set).

THEOREM 3. *Let Λ be as in Theorem 2. Then $C_\Lambda(T)$ is locally a Rosenthal set and has locally the Schur property, but $C_\Lambda(T)$ has neither of these properties.*

Proof. By Theorem 2 the result is obvious for the Schur property. If $\Lambda \cap v(k) = \{k\}$, clearly

$$C_{\Lambda \cap v(k)}(T) = L^\infty_{\Lambda \cap v(k)}(T).$$

By Theorem 2, $C_\Lambda(T)$ (hence $L^\infty_\Lambda(T)$) has a closed subspace isomorphic to c_0. By [1], $L^\infty_\Lambda(T)$, which is a dual space, has a closed subspace isomorphic to l^∞; hence $L^\infty_\Lambda(T)$ cannot be the same space as the separable space $C_\Lambda(T)$ and Λ is not a Rosenthal set.

By using the same example we can solve Problem 1 of [6].

DEFINITION 4 ([6], Lemma 1, Definition 4, Eberlein theorem). A function $F \in L^\infty(T)$ is *totally ergodic* if there exists a net $(v_\alpha) \in l^1(T_d)$ such that

(i) $\|v_\alpha\|_{l^1(T_d)} = \langle v_\alpha, 1 \rangle = 1$;

(ii) $\hat{v}_\alpha$ is supported by $v_\alpha(0)$ and $(v_\alpha(0))_\alpha$ is a basis of neighborhoods of $\{0\}$ in $\bar{Z}$.

(iii) $\forall k \in Z$ $(e^{-2i\pi kt}F) * v_\alpha \to \hat{F}(k)$ uniformly on T.

Every continuous function on T is totally ergodic. Hence, if Λ is a Rosenthal set, every function $F \in L^\infty_\Lambda(T)$ is totally ergodic. The converse does not hold since we have

THEOREM 4. *Let Λ be as in Theorem 2. Every function in $L^\infty_\Lambda(T)$ is totally ergodic but Λ is not a Rosenthal set.*

Proof. We have already proved in Theorem 3 that Λ is not a Rosenthal set. For every $k \in \boldsymbol{Z}$ let $v(k)$ be as in the proof of Theorem 2. Let

$$G_\alpha = (e^{-2i\pi k \cdot} F) * \nu_\alpha.$$

Hence

$$\hat{G}_\alpha(n) = \hat{F}(k+n)\hat{\nu}_\alpha(n) \quad \text{for every } n \in \boldsymbol{Z}.$$

As soon as $k + \nu_\alpha(0) \subset v(k)$ we have $\hat{G}_\alpha(n) = 0$ if $n \neq 0$ and $\hat{G}_\alpha(0) = \hat{F}(k)$ (two cases must be considered: either $v(k) \cap \Lambda = \emptyset$ or $v(k) \cap \Lambda = \{k\}$). This means that G_α is the constant function $\hat{F}(k)$ and that F is totally ergodic.

3. We now generalize Theorem 1 by using an idea of [5].

THEOREM 5. *Let $\Lambda = (\lambda_j)_{j \geqslant 1} \subset \boldsymbol{Z}$ be a set of first kind. We assume that*

$$f_n(t) = n^{-1} \sum_{j=1}^{n} \exp(2i\pi\lambda_j t)$$

converges pointwise on $\boldsymbol{T}$ to $l \in c_0(\boldsymbol{T})$. If $\Lambda' \subset \Lambda$ has a positive upper density with respect to Λ, then $C_{\Lambda'}(\boldsymbol{T})$ has a closed subspace isomorphic to c_0.

In Theorem 3.1 of [5] or Theorem 3 of [6] we only considered the case

$$\Lambda = \boldsymbol{Z}, \quad f_n(t) = (2n+1)^{-1} \sum_{j=-n}^{n} \exp(2i\pi jt);$$

hence $l(t) = 0$ for $t \neq 0$ and $l(0) = 1$.

Proof. By assumption there exists a subsequence $(n_k)_{k \geqslant 1}$ such that

$$n_k^{-1}\delta(n_k) \to \delta,$$

where δ is the positive upper density of Λ' with respect to Λ. We then have to consider the subsequence $(f_{n_k})_{k \geqslant 1}$. We will however write $n \geqslant 1$ instead of $(n_k)_{k \geqslant 1}$ in order to simplify the notation.

By Proposition 3.1 of [5], Theorem 5 will be proved if we can exhibit a non-zero function $l' \in c_0(\boldsymbol{T}_d)$ and a uniformly bounded net (f'_α) in $C_{\Lambda'}(\boldsymbol{T})$ such that

$$\langle f'_\alpha, \mu \rangle \to \langle l', \mu \rangle \quad \text{for every } \mu \in M(\boldsymbol{T}).$$

Let us define:

$$\nu_n = n^{-1} \sum_{1 \leqslant j \leqslant n, \lambda_j \in \Lambda} \delta_{\lambda_j},$$

whence $\nu_n \in l^1(\boldsymbol{Z})$ and $\hat{\nu}_n = f_n$;

$$\nu'_n = 1_{\Lambda'} \cdot \nu_n = n^{-1} \sum_{1 \leqslant j \leqslant n, \lambda_j \in \Lambda'} \delta_{\lambda_j},$$

whence $\nu'_n \in l^1(\boldsymbol{Z})$;

$$f'_n(t) = n^{-1} \sum_{1 \leqslant j \leqslant n, \lambda_j \in \Lambda'} \exp(2i\pi\lambda_j t),$$

whence $\hat{\nu}'_n = f'_n$. Then

(i) $\|\nu_n\|_{l^1(\boldsymbol{Z})} = \langle \nu_n, 1 \rangle = 1 = f_n(0)$,

(ii) $\forall t \in T\ \hat{v}_n(t) \to l(t)$,

(iii) $\|v'_n\|_{l^1(Z)} = \langle v'_n, 1\rangle = f'_n(0) = n^{-1}\delta(n) \leqslant 1$ and $n^{-1}\delta(n) \to \delta$.

By (i) and (ii) there exists a positive measure $v \in M(\bar{Z})$ such that

$$\hat{v} = l \quad \text{and} \quad v_n \to v \ \sigma(M(\bar{Z}), C(\bar{Z})).$$

Obviously, v is supported by $\bar{\Lambda}$.

Let us show that $\langle f'_n, \mu\rangle \to 0$ for every diffuse measure $\mu \in M(T)$:

$$|\langle f'_n, \mu\rangle| = |\langle v'_n, \hat{\mu}\rangle| = n^{-1}\delta(n)|\langle v'_n \|v'_n\|^{-1}_{l^1(Z)}, \hat{\mu}\rangle|$$
$$\leqslant (n^{-1}\delta(n))^{1/2}\langle v'_n, |\hat{\mu}|^2\rangle^{1/2} \leqslant \langle v_n, |\hat{\mu}|^2\rangle^{1/2} = \langle f_n, \mu*\check{\mu}\rangle^{1/2}$$

(where $\check{\mu}(E) = \bar{\mu}(-E)$ for every Borel set $E \subset T$). By assumption,

$$\langle f_n, \mu*\check{\mu}\rangle \to \langle l, \mu*\check{\mu}\rangle = 0.$$

Let now $v' \in M(\bar{Z})$ be a limit point of $(v'_n)_{n\geqslant 1}$ for $\sigma(M(\bar{Z}), C(\bar{Z}))$, i.e.,

$$\langle v'_\alpha, \varphi\rangle \to \langle v', \varphi\rangle \quad \text{for every } \varphi \in C(\bar{Z}).$$

Obviously, v' is supported by $\bar{\Lambda}'$.

Let us define

$$l'(t) = \hat{v}'(t) = \lim_\alpha \hat{v}'_\alpha(t) = \lim_\alpha f'_\alpha(t) \qquad (t \in T).$$

We shall prove that $l' \in c_0(T_d)$. As a consequence, $\langle l', \mu\rangle = 0$ for every diffuse measure μ on T; hence

$$\langle l', \mu\rangle = 0 = \lim_\alpha \langle f'_\alpha, \mu\rangle,$$

and for every $\mu \in M(T)$

$$\langle l', \mu\rangle = \lim_\alpha \langle f'_\alpha, \mu\rangle.$$

We first prove that v' defines a continuous linear form on $L^1(v)$. For every $\varphi \in C(\bar{Z})$

$$|\langle v', \varphi\rangle| = \lim_\alpha |\langle v'_\alpha, \varphi\rangle| \leqslant \lim_\alpha \langle v'_\alpha, |\varphi|\rangle$$
$$\leqslant \lim_\alpha \langle v_\alpha, |\varphi|\rangle = \langle v, |\varphi|\rangle = \|\varphi\|_{L^1(v)}.$$

In particular, v' is absolutely continuous with respect to v, i.e., $v' \in L^1(v)$; hence there exists a sequence $(g_k)_{k\geqslant 1}$ in $C(\bar{Z})$ such that

$$\|g_k v - v'\|_{M(\bar{Z})} \to 0$$

and we may assume that every $\hat{g}_k$ is finitely supported on T. A fortiori,

$$\|\widehat{g_k v} - l'\|_{l^\infty(T_d)} \to 0.$$

As $l \in c_0(T_d)$ and $\widehat{g_k v} = \hat{g}_k * l$, we have proved that $l' \in c_0(T_d)$.

Remark 1. Let $\Lambda \subset Z$; let us recall that $\bar{\Lambda}$ is an M_0-*set* in $\bar{Z}$ if there exists a measure $\nu \in M(\bar{Z})$, supported by $\bar{\Lambda}$, such that $\hat{\nu} \in c_0(T_d)$. Under the assumptions of Theorem 5 we have proved that $\bar{\Lambda}$ and $\bar{\Lambda}'$ are M_0-sets in $\bar{Z}$. In particular, the sets $\bar{P}$ and $\overline{\{j^s\}_{j\geqslant 1}}$ $(s \in N \backslash 0)$ are M_0-sets in $\bar{Z}$. By [9], $\bar{P}$ is a set of Haar measure 0.

We could not answer the following question:

Let $\Lambda \subset Z$ be such that $\bar{\Lambda}$ is an M_0-set. Does $C_\Lambda(T)$ have a closed subspace isomorphic to c_0? **(P 1374)**

We now give a sufficient condition in order that $\Lambda' = \Lambda \cap v(k)$ has a positive upper density with respect to Λ when Λ is a set of first kind (Q), $k \in Z$ and $v(k)$ is any neighborhood of k in $\bar{Z}$. Obviously, k must be in $\bar{\Lambda}$. We generalize the method used in [4] (Lemma 3 and the proof of Theorem 4) for $k = 0$ and $\Lambda = \{j^s\}_{j\geqslant 1}$ on one hand, for $k = +1$ and $\Lambda = P$ on the other hand.

THEOREM 6. *Let $\Lambda = \{\lambda_j\}_{j\geqslant 1} \subset Z$ be a set of first kind (Q). Let $k \in Z$ be such that, for every $R = k+qZ$ $(q \geqslant 1)$, $\Lambda \cap R$ has a positive upper density with respect to Λ. Then $k \in \bar{\Lambda}$ and, for every neighborhood $v(k)$ in $\bar{Z}$, $\Lambda \cap v(k)$ has a positive upper density with respect to Λ.*

Proof. For every $k \in Z$ and every $v(k)$ there exist ([4], Lemma 3) an integer $q \geqslant 1$, irrational numbers $\beta_1, \ldots, \beta_L$ which are independent over Q and $\delta > 0$ such that $v(k)$ contains the set

$$E_k = \{n \in Z \mid n \in k+qZ = R \text{ and } |\exp(2i\pi n\beta_l q^{-1}) - \exp(2i\pi k\beta_l q^{-1})| < \delta,\ 1 \leqslant l \leqslant L\}.$$

By assumption and Theorem 1(a) the sequence $(\lambda_j t)_{j\geqslant 1, \lambda_j \in \Lambda\cap R}$ is uniformly distributed for every $t \notin Q$. Let us put $\Lambda \cap R = \{\lambda_{j_m}\}_{m\geqslant 0}$. Thus for every $h_1, \ldots, h_L \in Z$ not all zero we have

$$r^{-1} \sum_{m=1}^{r} \exp\Big(2i\pi\lambda_{j_m} \sum_{l=1}^{L} h_l \beta_l q^{-1}\Big) \to 0 \qquad (r \to +\infty).$$

Let

$$(t_m^{(l)})_{l=1}^{L} = (\lambda_{j_m} \beta_l q^{-1})_{l=1}^{L} \in T^L \quad \text{and} \quad \mu_r = r^{-1} \sum_{m=1}^{r} \delta_{t_m^{(1)}, \ldots, t_m^{(l)}}.$$

We have just proved that for every $f \in C(T^L)$

$$\langle \mu_r, f \rangle \to \int f dt^{(1)} \ldots dt^{(L)}.$$

By taking a non-zero f with values in $[0, 1]$, supported by a suitably chosen neighborhood $(k\beta_l q^{-1})_{l=1}^{L} \in T^L$ we see that $\Lambda \cap E_k$ has a positive density with respect to $\Lambda \cap R$, and hence a positive upper density with respect to Λ.

THEOREM 7. (a) *Let P be the set of prime numbers. For every neighborhood $v(+1)$ in $\bar{Z}$, $C_{P\cap v(1)}(T)$ has a closed subspace isomorphic to c_0. The same is true for every $v(-1)$.*

(b) *Let* $\Lambda = \{j^s\}_{j \geqslant 1}$. *If* s *is an even integer,*

$$\bar{\Lambda} \cap \boldsymbol{Z} = \Lambda \cup \{0\}.$$

If s *is an odd integer,*

$$\bar{\Lambda} \cap \boldsymbol{Z} = \Lambda \cup (-\Lambda) \cup \{0\}.$$

For every $k \in \bar{\Lambda}$ *and every neighborhood* $v(k)$ *in* $\bar{\boldsymbol{Z}}$, $\Lambda \cap v(k)$ *has a positive upper density with respect to* Λ *and* $C_{\Lambda \cap v(k)}(T)$ *has a closed subspace isomorphic to* c_0.

Proof. (a) $P \cap v(1)$ or $P \cap v(-1)$ has a positive upper density with respect to P by the proof of Theorem 4 in [4] or by Theorem 6 and the properties of $P \cap R$ recalled in Example 1. We conclude by Theorem 5.

(b) The fact that $\overline{\{j^2\}_{j \geqslant 0}} \cap \boldsymbol{Z} = \{j^2\}_{j \geqslant 0}$ is proved in Lemma 3.6.2 of [3] by considering two cases: $k < 0$ and $k \in \boldsymbol{N} \backslash \Lambda$. We follow the same method. Let s be an integer $\geqslant 2$ and $k \in \boldsymbol{N} \backslash \Lambda$, $k \neq 0$. There exist a prime number p, $n \geqslant 0$, n', $k' \geqslant 1$ such that $k = p^{sn+n'}k'$, $1 \leqslant n' < s$, and p does not divide k'. Let

$$R = k + p^{s(n+1)}\boldsymbol{Z};$$

then $\Lambda \cap R$ is empty. Let s be $\geqslant 2$ and $k < 0$. Let

$$R = k + 3|k|^s \boldsymbol{Z}.$$

If $\Lambda \cap R$ is not empty, there exist $j \in \boldsymbol{N}$ and $q \in \boldsymbol{Z}$ such that

$$j^s = k(1 - 3|k|^{s-1}q) = |k|\,\big|-1 + 3|k|^{s-1}q\big|.$$

These two numbers are coprime, and hence $|k|$, $\big|-1+3|k|^{s-1}q\big| \in \Lambda$. If s is an even integer, this is impossible because the equation $-1 \equiv j^2(3)$ has no solution. (This is taken from [8].) Hence

$$\bar{\Lambda} \cap \boldsymbol{Z} \subset \Lambda \cup \{0\}.$$

If s is odd, this implies $k \in -\Lambda$. Hence

$$\bar{\Lambda} \cap \boldsymbol{Z} \subset \Lambda \cup (-\Lambda) \cup \{0\}.$$

If $k \in \Lambda \cup \{0\}$ and $s \geqslant 1$, $\Lambda \cap R$ is not empty for every $R = k + q\boldsymbol{Z}$ $(q \geqslant 1)$ and has a positive upper density with respect to Λ as it was recalled in Example 2. If $k \in -\Lambda$ and s is odd, for every $R = k + q\boldsymbol{Z} = -j_0^s + q\boldsymbol{Z}$ let $a \in \boldsymbol{N}$ be such that $aq - j_0 > 0$. Then

$$(aq - j_0)^s \in \Lambda \cap R;$$

hence $\Lambda \cap R$ has a positive upper density with respect to Λ. In both cases, by Theorem 6, $v(k) \cap \Lambda$ is not empty for every $v(k)$; hence $k \in \bar{\Lambda}$. Theorem 6 again and Theorem 5 now conclude the proof.

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