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# CONVERGENCE RATES FOR WEIGHTED SUMS OF RANDOM VARIABLES WITH RANDOM INDICES

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**1. Introduction and preliminaries.** Let $\{a_{ik}\}$ and $\{b_{ik}\}$ for $i, k = 1, 2, \ldots$ be double sequences of real numbers and let $\{X_k, k \geqslant 1\}$ be a sequence of not necessarily identically distributed random variables, defined on a probability space $(\Omega, \mathscr{A}, \mathrm{P})$. Limit properties of sums

$$S_i = \sum_{k=1}^{\infty} a_{ik}(X_k - b_{ik}) \quad \text{as } i \to \infty \tag{1}$$

have been investigated in various papers, e.g. in [1]-[3] and [5]-[8]. It was proved that the sums of form (1) have many properties similar to those for random variables

$$T_i = \frac{1}{i} \sum_{k=1}^{i} X_k \quad \text{as } i \to \infty. \tag{2}$$

The aim of this paper is to give sufficient conditions for convergence in probability to zero of sums

$$S_{N_n} = \sum_{k} a_{N_n k}(X_k - b_{N_n k}) \quad \text{as } n \to \infty, \tag{3}$$

where $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables defined on the same probability space $(\Omega, \mathscr{A}, \mathrm{P})$.

We assume that for every $n \geqslant 1$ and $k \geqslant 1$ the random variables $N_n$ and $X_k$ are independent. Moreover, we suppose that sequences $\{N_n, n \geqslant 1\}$ and $\{a_{ik}, i, k \geqslant 1\}$ are such that either

$$\mathrm{E} \sup_j |a_{N_n j}| \sum_k |a_{N_n k}|^t = o\Big(\mathrm{E} \sum_k |a_{N_n k}|^t\Big) \quad \text{as } n \to \infty \tag{4}$$

or

$$\mathrm{E}\Big(\sum_k |a_{N_n k}|^t\Big)^2 = o\Big(\mathrm{E} \sum_k |a_{N_n k}|^t\Big) \quad \text{as } n \to \infty, \tag{5}$$

where $0 < t < \infty$.

Further on, we suppose that $\{\varrho_i(t), i \geqslant 1\}$, $t \in R$, is a sequence of non-negative real numbers such that

$$\mathrm{E}\sum_k |a_{N_n k}|^t \leqslant \mathrm{E}\varrho_{N_n}(t), \quad n \geqslant 1. \tag{6}$$

To abbreviate the notation, set

$$F_k(y) = \mathrm{P}[X_k < y], \quad F'_k(y) = \mathrm{P}[X_k - \mathrm{E}X_k < y], \quad k \geqslant 1,$$

if $\mathrm{E}X_k$ exists, and put

$$F(y) = \sup_k \mathrm{P}[|X_k| \geqslant y], \quad F'(y) = \sup_k \mathrm{P}[|X_k - \mathrm{E}X_k| \geqslant y].$$

**2. Results.**

THEOREM 1. *Let $\{X_k, k \geqslant 1\}$ be a sequence of random variables and $\{N_n, n \geqslant 1\}$ a sequence of positive integer-valued random variables both defined on a probability space $(\Omega, \mathscr{A}, \mathrm{P})$ and such that, for every $k \geqslant 1$ and $n \geqslant 1$, $X_k$ and $N_n$ are independent.*

1° *If $0 < t < 1$, $y^t F(y) \leqslant M < \infty$ for all $y \geqslant 0$, and* (6) *holds, then*

$$\mathrm{P}[|S_{N_n}| > \varepsilon] = O(\mathrm{E}\varrho_{N_n}(t)) \quad \text{as } n \to \infty, \tag{O}$$

*with $b_{N_n k} = 0$ a.s.*

2° *If $0 < t < 1$, $y^t F(y) \to 0$ as $y \to \infty$, and* (4) *and* (6) *hold, then*

$$\mathrm{P}[|S_{N_n}| > \varepsilon] = o(\mathrm{E}\varrho_{N_n}(t)) \quad \text{as } n \to \infty, \tag{o}$$

*with $b_{N_n k} = 0$ a.s.*

In the following theorems we assume that $\{X_k, k \geqslant 1\}$ is a sequence of independent random variables, and a sequence $\{N_n, n \geqslant 1\}$ satisfies the assumptions of Theorem 1.

THEOREM 2. 1° *If $t = 1$, $yF(y) \leqslant M < \infty$ for all $y \geqslant 0$, inequality* (6) *is satisfied, and*

$$\overline{\lim_{T \to \infty}} \sup_k \Big| \int_{-T}^{T} y\, dF_k(y) \Big| < \infty, \tag{7}$$

*then* (O) *holds with $b_{N_n k} = 0$ a.s.*

2° *If $t = 1$, $yF(y) \to 0$ as $y \to \infty$, and* (5), (6), *and* (7) *are satisfied, then* (o) *holds with $b_{N_n k} = 0$ a.s.*

THEOREM 3. 1° *If $t = 1$, $yF(y) \leqslant M < \infty$, and* (6) *is satisfied, then* (O) *holds with*

$$(8)\qquad b_{N_nk} = \begin{cases} \int\limits_{-|a_{N_nk}|^{-1}}^{|a_{N_nk}|^{-1}} y dF_k(y) & \text{if } N_n \notin [i: a_{ik} = 0], \\ 0 & \text{if } N_n \in [i: a_{ik} = 0] \text{ a.s.} \end{cases}$$

2° *If* $t = 1$, $yF(y) \to 0$ *as* $y \to \infty$, *and* (4) *and* (6) *are satisfied, then* (o) *holds with* $b_{N_nk}$ *given by* (8).

THEOREM 4. 1° *If* $1 < t < 2$, $y^tF(y) \leqslant M < \infty$, *and* (6) *is satisfied, then* (O) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

2° *If* $1 < t < 2$, $y^tF(y) \to 0$ *as* $y \to \infty$, *and* (4) *and* (6) *are satisfied, then* (o) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

THEOREM 5. 1° *If* $t = 2$, $y^2F(y) \leqslant M < \infty$, *inequality* (6) *is satisfied and, moreover,*

$$\exists\, 0 < \lambda < \infty \ \forall\, 0 < \varrho, \mu \leqslant \max[8, 4/\lambda]+2,$$

$$(9)\qquad \mathrm{E}\Big(\sum_k a_{N_nk}^2\Big)^{\varrho}\Big(\sum_j a_{N_nj}^2 |\log|a_{N_nj}||\Big)^{\mu} = O\Big(\mathrm{E}\Big(\sum_k a_{N_nk}^2\Big)^{\varrho+\lambda\mu}\Big) \quad \text{as } n \to \infty,$$

*then* (O) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

2° *If* $t = 2$, $y^2F(y) \to 0$ *as* $y \to \infty$, *and* (5), (6), *and* (9) *are satisfied, then* (o) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

THEOREM 6. 1° *If* $t > 2$, $y^tF(y) \leqslant M < \infty$, *inequality* (6) *is satisfied and, moreover,*

$$\exists\, 0 < a < \infty \ \forall\, a < \beta < 6ta+2 \ \forall\, 0 < \gamma < 6ta+2,$$

$$(10)\qquad \mathrm{E}\Big(\sum_k a_{N_nk}^2\Big)^{\beta}\Big(\sum_k |a_{N_nk}|^t\Big)^{\gamma} = O\Big(\mathrm{E}\Big(\sum_k |a_{N_nk}|^t\Big)^{\beta/a+\gamma}\Big)$$

*as* $n \to \infty$, *then* (O) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

2° *If* $t > 2$, $y^tF(y) \to 0$ *as* $y \to \infty$, *and* (5), (6), *and* (10) *are satisfied, then* (o) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

LEMMA 1. *The following conditions are equivalent:*

$$(\mathrm{A})\qquad \begin{cases} \textit{There exists a random variable } X \textit{ such that} \\ (\mathrm{a}_1)\ \forall\, k \in N,\ \mathrm{P}[|X_k| \geqslant y] \leqslant \mathrm{P}[|X| \geqslant y], \\ (\mathrm{a}_2)\ \mathrm{E}|X|^t \leqslant M_1 < \infty. \end{cases}$$

$$(\mathrm{B})\qquad \begin{cases} (\mathrm{b}_1)\ F(y) \to 0 \textit{ as } y \to \infty, \\ (\mathrm{b}_2)\ \int\limits_0^{\infty} y^t |dF(y)| \leqslant M_2 < \infty. \end{cases}$$

THEOREM 7. *If* $t = 1$, $F(y) \to 0$ *as* $y \to \infty$, (4) *and* (6) *are satisfied and, moreover,*

$$\int_0^\infty y\,|dF(y)| \leqslant M < \infty, \tag{11}$$

*then* (o) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

By Lemma 1, if $\{X_k, k \geqslant 1\}$ is a sequence of independent random variables uniformly bounded by a random variable $X$ such that $\mathrm{E}|X| < \infty$, we have

THEOREM 7'. *If* $t = 1$, *and* (4) *and* (6) *are satisfied, then* (o) *holds with* $b_{N_nk} = \mathrm{E}X_k$ *a.s.*

In all proofs we put $\mathrm{P}[N_n = i] = p_{in}$. Summations in (1) and (3) may be taken only over those values of $i$ for which $a_{ik} = 0$. Integrals will be Lebesgue-Stieltjes ones. By $C$ we shall denote different, in general, positive absolute constants.

Proof of Theorem 1. By Theorem 1 of [8], for any given $\varepsilon > 0$ we have

$$\mathrm{P}[|S_i| > \varepsilon] \leqslant C \sum_k |a_{ik}|^t,$$

and since $X_k$ and $N_n$ are independent for any $k \geqslant 1$ and $n \geqslant 1$, we see that 1° holds.

To prove 2° let us put

$$I = [i: \sup_k |a_{ik}| \leqslant \eta],$$

where $\eta > 0$ will be fixed later. Then for an arbitrary $\varepsilon > 0$ we have

$$\mathrm{P}[|S_{N_n}| > \varepsilon] = \sum_{i\in I} \mathrm{P}[|S_i| > \varepsilon]p_{in} + \sum_{j\notin I} \mathrm{P}[|S_j| > \varepsilon]p_{jn}.$$

Now let $\tau$ be any given positive number. For sufficiently small fixed $\eta$, by Theorem 2 of [8] we obtain

$$\sum_{i\in I} \mathrm{P}[|S_i| > \varepsilon]p_{in} \leqslant \frac{\tau}{2} \mathrm{E}\,\varrho_{N_n}(t).$$

Further, by (4), for sufficiently large $n$ we have

$$\sum_{j\notin I} \mathrm{P}[|S_j| > \varepsilon]p_{jn} \leqslant \frac{1}{\eta^2} \sum_{j\notin I} \sup_l |a_{jl}| \sum_k |a_{jk}|^t p_{jn} \leqslant \frac{\tau}{2} \mathrm{E}\varrho_{N_n}(t),$$

which proves 2°.

Using Theorems 1b and 2b of [3] and the considerations given in the proof of Theorem 1 one can easily establish the statements of Theorem 2.

Proof of Theorem 3. 1° Since random variables $X_k$ are independent of $N_n$, we have the inequality

$$(12)\qquad P[|S_{N_n}| > \varepsilon] \leqslant \sum_i \sum_k P[|a_{ik}X_k| > 1]p_{in} + \\ + \sum_i P\Big[\Big|\sum_k a_{ik}(Y_{ik} - b_{ik})\Big| > \varepsilon\Big] p_{in},$$

where $Y_{ik} = X_k \mathscr{I}[|a_{ik}X_k| \leqslant 1]$, and $\mathscr{I}[A]$ denotes the indicator of $A$.

The first term on the right-hand side of (12) can be bounded as follows:

$$(13)\qquad \sum_i \sum_k P[|a_{ik}X_k| > 1]p_{in} \leqslant \sum_i \sum_k F\left(\frac{1}{|a_{ik}|}\right) p_{in} \\ \leqslant M \sum_i \sum_k |a_{ik}|^t p_{in} = O(\mathrm{E}\varrho_{N_n}(t)).$$

For the second term in (12), since $b_{ik} = \mathrm{E}Y_{ik}$, by Chebyshev's inequality we have the estimation

$$(14)\qquad \sum_i P\Big[\Big|\sum_k a_{ik}(Y_{ik} - b_{ik})\Big| > \varepsilon\Big] p_{in} \\ \leqslant C \sum_i \sum_k a_{ik}^2 \mathrm{E} Y_{ik}^2 p_{in} \leqslant C \sum_i \sum_k \int_0^{|a_{ik}|^{-1}} 2yF(y)\,dy\, p_{in} \\ \leqslant C \sum_i \sum_k |a_{ik}| p_{in} = O(\mathrm{E}\varrho_{N_n}(t)).$$

To prove statement (o) we make the assumption

$$\forall \tau > 0\ \exists \eta > 0\ \forall y \geqslant \frac{1}{\eta},\quad y^t F(y) \leqslant \frac{\tau}{2},$$

estimating the terms on the right-hand side of (12). It can be seen that, for

$$i \in I = [i: \sup_k |a_{ik}| \leqslant \eta],$$

we have

$$(15)\qquad \sum_i \sum_k P[|a_{ik}X_k| > 1]p_{in} \leqslant \sum_i \sum_k F\left(\frac{1}{|a_{ik}|}\right) p_{in} \leqslant \frac{\tau}{2} \mathrm{E}\varrho_{N_n}(t).$$

If $j \notin I$, then, using (4),

$$(16)\qquad \sum_j \sum_k P[|a_{jk}X_k| > 1]p_{jn} \\ \leqslant C \sum_j \sup_k |a_{jk}| \sum_l |a_{jl}|^t \sup_{y \geqslant 0} y^t F(y) p_{jn} \leqslant \frac{\tau}{2} \mathrm{E}\varrho_{N_n}(t).$$

Now, let us observe that

$$\forall \tau > 0 \ \exists \eta > 0 \ \forall T \geqslant \frac{1}{\eta}, \ \frac{1}{T}\int_0^T 2yF(y)\,dy \leqslant \frac{\tau\varepsilon^2}{2}.$$

By this fact and considerations which have led to (14), we have, for $i \in I$,

$$\sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Y_{ik}-b_{ik})\Big| > \varepsilon\Big]p_{in} \leqslant \frac{\tau}{2}\mathrm{E}\varrho_{N_n}(t). \tag{17}$$

If $j \notin I$, then by (4) we get

$$\sum_j \mathrm{P}\Big[\Big|\sum_k a_{jk}(Y_{jk}-b_{jk})\Big| > \varepsilon\Big]p_{jn} \leqslant C\sum_j \sup_l |a_{jl}| \sum_k |a_{jk}|p_{jn} \sup_{T>0}\frac{1}{T}\int_0^T 2yF(y)\,dy \leqslant \frac{\tau}{2}\mathrm{E}\varrho_{N_n}(t). \tag{18}$$

Inequalities (15)-(18) together complete the proof of 2°.

Proof of Theorem 4. By Theorem 1c of [3] we know that

$$\mathrm{P}[|S_i| > \varepsilon] \leqslant C\sum_k |a_{ik}|^t$$

and we see that the first assertion of Theorem 4 can be obtained in the same way as 1° of Theorem 1.

To prove 2° we need the inequality

$$\begin{aligned}\sum_i \mathrm{P}[|S_i| > 2\varepsilon]p_{in} &\leqslant \sum_i\sum_k \mathrm{P}[|a_{ik}(X_k-\mathrm{E}X_k)| > 1]p_{in} \\ &\leqslant \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big]p_{in} + \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]p_{in},\end{aligned} \tag{19}$$

where

$$Z_{ik} = (X_k-\mathrm{E}X_k)\mathcal{I}[|a_{ik}(X_k-\mathrm{E}X_k)| \leqslant 1].$$

Taking into account (15) and (16) and putting $F'$ instead of $F$ we see that

$$\sum_i \mathrm{P}[|a_{ik}(X_k-\mathrm{E}X_k)| > 1]p_{in} = o\big(\mathrm{E}\varrho_{N_n}(t)\big).$$

Now we observe that

$$|\mathrm{E}Z_{ik}| \leqslant \int_{|a_{ik}|^{-1}}^{\infty} y\,|dF'(y)| = \frac{1}{|a_{ik}|}F'\Big(\frac{1}{|a_{ik}|}\Big) + \int_{|a_{ik}|^{-1}}^{\infty} F'(y)\,dy. \tag{20}$$

Using (20), we have

$$(21)\quad \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big]p_{in}$$

$$\leqslant C\sum_i\sum_k F'\Big(\frac{1}{|a_{ik}|}\Big)p_{in} + C\sum_i\sum_k |a_{ik}|\int_{|a_{ik}|^{-1}}^{\infty} F'(y)\,dy.$$

By the same arguments as for (15) and (16), we get

$$\sum_i\sum_k F'\Big(\frac{1}{|a_{ik}|}\Big)p_{in} = o\big(\mathrm{E}\varrho_{N_n}(t)\big).$$

Since, by the assumptions of Theorem 4,

$$\forall\,\tau > 0\ \exists\,\eta > 0\ \forall\,|a_{ik}| \leqslant \eta,\quad \int_{|a_{ik}|^{-1}}^{\infty} F'(y)\,dy \leqslant \frac{\tau}{2}\varepsilon|a_{ik}|^{t-1},$$

for $i \in I$ we have

$$(22)\quad \sum_i\sum_k |a_{ik}|\int_{|a_{ik}|^{-1}}^{\infty} F'(y)\,dy\,p_{in} \leqslant \frac{\tau}{2}\mathrm{E}\varrho_{N_n}(t).$$

If $j \notin I$, then by (4)

$$(23)\quad \sum_j\sum_k |a_{jk}|\int_{|a_{jk}|^{-1}}^{\infty} F'(y)\,dy\,p_{jn} \leqslant C\sum_j\sum_k |a_{jk}|\int_{|a_{jk}|^{-1}}^{\infty} y^{-t}\,dy\,p_{jn}$$

$$\leqslant C\sum_j\sum_k |a_{jk}|^t\sup_l|a_{jl}|\,p_{jn} \leqslant \frac{\tau}{2}\mathrm{E}\varrho_{N_n}(t).$$

To estimate the last term of (19), let us first observe that under the assumptions of Theorem 4

$$\forall\,\tau > 0\ \exists\,\eta > 0\ \forall\,y \geqslant \frac{1}{\eta},\quad 2yF'(y) \leqslant \tau y^{1-t}(2-t).$$

Now set

$$I = \big[i\colon \sup_k|a_{ik}| \leqslant \min[\eta,\ \tau^{1/(2-t)}]\big].$$

If $i \in I$, then

$$(24)\quad \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]p_{in} \leqslant \frac{1}{\varepsilon^2}\sum_i\sum_k a_{ik}^2\int_0^{|a_{ik}|^{-1}} 2yF'(y)\,dy\,p_{in}$$

$$\leqslant C\sum_i\sum_k a_{ik}^2\int_0^{1/\eta} 2y\,dy\,p_{in} + \tau C\sum_i\sum_k a_{ik}^2|a_{ik}|^{t-2}p_{in}$$

$$\leqslant C\sum_i\sum_k |a_{ik}|^t\sup_l|a_{il}|^{2-t}p_{in} + \tau C\sum_i\sum_k |a_{ik}|^t p_{in} \leqslant \tau C\mathrm{E}\varrho_{N_n}(t).$$

If $j \notin I$, then, by (4),

$$\sum_j \mathrm{P}\Big[\Big|\sum_k a_{jk}(Z_{jk}-\mathrm{E}Z_{jk})\Big| > \varepsilon\Big] p_{jn} \leqslant C \sum_j \sum_k a_{jk}^2 \int_0^{|a_{jk}|^{-1}} y^{1-t}\, dy\, p_{jn} \tag{25}$$

$$\leqslant C \sum_j \sum_k |a_{jk}|^t \sup |a_{jl}| p_{jn} \leqslant \tau \mathrm{E}\varrho_{N_n}(t).$$

Thus we have proved Theorem 4.

Proof of Theorem 5. First we observe that, by the inequality given in [1],

$$\mathrm{P}[|S_{N_n}| > 3\varepsilon] \leqslant \sum_i \sum_k \mathrm{P}[|a_{ik}(X_k - \mathrm{E}X_k)| > \varepsilon] p_{in} + \tag{26}$$

$$+ \sum_i \sum_{j \neq k} \mathrm{P}[|a_{ij}(X_j - \mathrm{E}X_j)| > \delta_i] \mathrm{P}[|a_{ik}(X_k - \mathrm{E}X_k)| > \delta_i] p_{in} +$$

$$+ \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big] p_{in} + \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik} - \mathrm{E}Z_{ik})\Big| > \varepsilon\Big] p_{in},$$

where

$$\delta_i = \Big(\sum_k a_{ik}^2\Big)^{1/4} \quad \text{and} \quad Z_{ik} = (X_k - \mathrm{E}X_k)\mathscr{I}[|a_{ik}(X_k - \mathrm{E}X_k)| \leqslant \delta_i].$$

Under the assumption $y^2 F(y) \leqslant M < \infty$ (see [3]) we have

$$\sum_i R_i p_{in} = \sum_i \sum_k \mathrm{P}[|a_{ik}(X_k - \mathrm{E}X_k)| > \varepsilon] p_{in} +$$

$$+ \sum_i \sum_{j \neq k} \mathrm{P}[|a_{ij}(X_j - \mathrm{E}X_j)| > \delta_i] \mathrm{P}[|a_{ik}(X_k - \mathrm{E}X_k)| > \delta_i] p_{in} +$$

$$+ \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big] p_{in} \leqslant C \sum_i \sum_k a_{ik}^2 p_{in}$$

and, consequently,

$$\sum_i R_i p_{in} \leqslant C \sum_i \sum_k a_{ik}^2 p_{in} = O\big(\mathrm{E}\varrho_{N_n}(t)\big). \tag{27}$$

Thus we only need to bound the last term in (26). We follow the method used in [1]-[3].

Let us choose a positive integer $\nu$ such that

$$\max[8, 4/\lambda] < 2\nu \leqslant \max[8, 4/\lambda] + 2.$$

Using the estimations given in the proof of Theorem 1d of [3], we obtain

(28) $$\sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]\, p_{in}$$

$$\leqslant \sum_i C \sum\nolimits^{*} \sum\nolimits^{**} \prod_{k=1}^{a} |a_{i\beta_k}|^{m_k} \mathrm{E}\,|Z_{i\beta_k}-\mathrm{E}Z_{i\beta_k}|^{m_k} p_{in}$$

$$\leqslant C \sum_i \sum\nolimits^{*} \delta_i^{2\nu-2a}\Big(\sum_k a_{ik}^2 \mathrm{E}Z_{ik}^2\Big)^a p_{in},$$

where the sum $\sum^{*}$ is taken over all integers $a, m_1, m_2, \ldots, m_a$ such that $2 \leqslant m_k$, $k = 1, 2, \ldots, a$, and $m_1+m_2+\ldots+m_a = 2\nu$, and in the sum $\sum^{**}$ subscripts $\beta_1, \beta_2, \ldots, \beta_a$ run over the positive integers.

It is enough to consider the case $\delta_i > e^{-2M'}$, where $y^2F'(y) \leqslant M' < \infty$ for all $y \geqslant 0$. Thus

$$\sum_k a_{ik}^2 \mathrm{E}Z_{ik}^2 \leqslant \sum_k a_{ik}^2\Big(2M'\log\frac{\delta_i}{|a_{ik}|}+1\Big) \leqslant 2M' \sum_k a_{ik}^2 \big|\log|a_{ik}|\big|.$$

Now by (9) and (28) we have

(29) $$\sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]\, p_{in}$$

$$\leqslant C \sum_i \sum\nolimits^{*} \Big(\sum_k a_{ik}^2\Big)^{(\nu-a)/2} \Big(\sum_k a_{ik}^2 \big|\log|a_{ik}|\big|\Big)^a p_{in}$$

$$\leqslant C \sum\nolimits^{*} \sum_i \Big(\sum_k a_{ik}^2\Big)^{(\nu-a)/a+\lambda a} p_{in}.$$

The sum $\sum^{*}$ is finite, since it depends only on $\nu$.

Now, we prove that for an arbitrary real number $\lambda > 0$ and for $a$, $0 < a \leqslant \nu$,

(30) $$\frac{\nu-a}{2}+\lambda a > 2.$$

If $\lambda \geqslant \frac{1}{2}$, then

$$\frac{\nu+a(2\lambda-1)}{2} > \frac{\nu}{2} > 2,$$

since $2\nu > \max[8, 4/\lambda]$. If $0 < \lambda < \frac{1}{2}$, then

$$\frac{\nu+a(2\lambda-1)}{2} \geqslant \frac{\nu+\nu(2\lambda-1)}{2} = \nu\lambda > 2.$$

Taking into account (30), we see that the last expression in (29) is $O(\mathrm{E}\varrho_{N_n}(t))$, which completes the proof of 1°.

To prove 2° let us put

$$I = \Big[i\colon \sum_k a_{ik}^2 < \eta\Big], \qquad \text{where } \eta > 0.$$

The consideration similar to that in the proof of Theorem 1, 2°, after using Theorem 2d of [3] and inequality (26), allows us to write

$$\sum_i R_i p_{in} = o\big(\mathrm{E}\varrho_{N_n}(t)\big).$$

Finally, by (28)-(30), we have proved 2°.

Proof of Theorem 6. 1° The first term on the right-hand side of (26) can be bounded as (13), changing $F$ into $F'$.

The second term is less than

$$\sum_i \Big(\sum_k |a_{ik}|^t \delta_i^{-t}\Big)^2 \Big\{\Big(\frac{\delta_i}{|a_{ik}|}\Big)^t F'\Big(\frac{\delta_i}{|a_{ik}|}\Big)\Big\}^2 p_{in}. \tag{31}$$

Now let us write

$$\delta_i = \max\Big[\Big(\sum_k |a_{ik}|^t\Big)^{1/3t}, \Big(\sum_k a_{ik}^2\Big)^{1/3t}\Big].$$

It suffices to consider the case where

$$\sum_k |a_{ik}|^t < 1.$$

Then

$$\delta_i^{-2t} \leqslant \Big(\sum_k |a_{ik}|^t\Big)^{-2t/3t} \leqslant \Big(\sum_k |a_{ik}|^t\Big)^{-1}.$$

Now (31) is of the form

$$\sum_i \sum_k |a_{ik}|^t \{\sup y^t F'(y)\}^2 = O\big(\mathrm{E}\varrho_{N_n}(t)\big),$$

sup being taken for

$$y \geqslant \Big(\sum_k |a_{ik}|^t\Big)^{-1/t}.$$

To bound the third term on the right-hand side of (26) we use the inequality

$$\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| \leqslant C\delta_i^{1-t}\sum_k |a_{ik}|^t \leqslant C\Big(\sum_k |a_{ik}|^t\Big)^{(1+2t)/3t} \tag{32}$$

(for the proof of this property, see [3]). If

$$C\Big(\sum_k |a_{ik}|^t\Big)^{(1+2t)/3t} < \varepsilon,$$

then

$$\mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big] = 0.$$

On the other hand, we have

$$(33)\qquad \sum_i P\Big[\Big|\sum_k a_{ik} \mathrm{E} Z_{ik}\Big| > \varepsilon\Big] p_{in} \leqslant C \sum_i \sum_k |a_{ik}|^t p_{in} = O\big(\mathrm{E}\varrho_{N_n}(t)\big).$$

Now we are going to estimate the last term of (26). Let us fix an integer $\nu$ such that $6ta < 2\nu \leqslant 6ta+2$, and a real number $\mu$, $0<\mu<t-2$. By Markov's inequality and by (2.29), (2.32), and (2.33) of [2], we get

$$(34)\qquad \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]$$

$$\leqslant C \sum\nolimits^{*} \sum\nolimits^{**} \prod_{k=1}^{a+b} |a_{i\beta_k}|^{m_k} \mathrm{E}|Z_{i\beta_k} - \mathrm{E}Z_{i\beta_k}|^{m_k}$$

$$\leqslant C \sum\nolimits^{*} \Big(\sum_k a_{ik}^2\Big)^{\frac{m_1+\ldots+m_a}{2}+\frac{b\mu}{t-2}} \Big(\sum_k |a_{ik}|^t\Big)^{\frac{b(t-\mu-2)}{t-2}} \delta_i^{m_{a+1}+\ldots+m_{a+b}-bt+b\mu},$$

where the sum $\sum^*$ is taken over all positive integers $a$, $b$ and $m_k$, $k=1,2,\ldots,a+b$, such that

$$2 \leqslant m_k < t \quad \text{for } k = 1,2,\ldots,a,$$

and

$$t \leqslant m_k \quad \text{for } k = a+1, a+2, \ldots, a+b \text{ and } m_1+m_2+\ldots+m_{a+b} = 2\nu,$$

and the sum $\sum^{**}$ is taken over all sets of positive integers $(\beta_1, \beta_2, \ldots, \beta_{a+b})$.

As previously, it suffices to consider the case where

$$\sum_k |a_{ik}|^t < 1.$$

Then

$$\delta_i = \Big(\sum_k a_{ik}^2\Big)^{1/3t}.$$

It is easy to verify that

$$(35)\qquad \Big(\sum_k a_{ik}^2\Big)^{\frac{m_1+\ldots+m_a}{2}} \delta_i^{-(m_1+\ldots+m_a)} \leqslant \Big(\sum_k a_{ik}^2\Big)^{(m_1+\ldots+m_a)\left(\frac{1}{2}-\frac{1}{3t}\right)}$$

and

$$(36)\qquad \Big(\sum_k a_{ik}^2\Big)^{\frac{\mu}{t-2}} \Big(\sum_k |a_{ik}|^t\Big)^{\frac{t-\mu-2}{t-2}} \delta_i^{-t} \leqslant \Big(\sum_k a_{ik}^2\Big)^{\frac{2}{3}\cdot\frac{\mu}{t-2}} \Big(\sum_k |a_{ik}|^t\Big)^{\frac{2}{3}\cdot\frac{t-\mu-2}{t-2}}.$$

Thus (34) takes the form

$$(37)\qquad \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big]$$

$$\leqslant C \sum\nolimits^{*} \Big(\sum_k a_{ik}^2\Big)^{(m_1+\ldots+m_a)\left(\frac{1}{2}-\frac{1}{3t}\right)+\frac{2b\mu}{3(t-2)}+\frac{2\nu+b\mu}{3t}} \Big(\sum_k |a_{ik}|^t\Big)^{\frac{5}{3}\cdot\frac{b(t-\mu-2)}{t-2}}.$$

By (10) and (37), we get

$$(38)\qquad \sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}(Z_{ik}-\mathrm{E}Z_{ik})\Big| > \varepsilon\Big] p_{in}$$
$$\leqslant C\sum_i \sum{}^{*}\Big(\sum_k |a_{ik}|^t\Big)^{\frac{m_1+\ldots+m_a}{a}\left(\frac{1}{2}-\frac{1}{3t}\right)+\frac{2}{3a}\cdot\frac{b\mu}{t-2}+\frac{2\nu}{3ta}+\frac{b\mu}{3ta}+\frac{5}{3}\frac{b(t-\mu-2)}{t-2}} p_{in}.$$

Since the sum $\sum^*$ is finite, it is enough to show that the exponent in (38) is greater than 1. This is true in view of the assumption $6ta < 2\nu \leqslant 6ta+2$. Thus we have proved 1°.

To prove 2° we estimate the right-hand side of (26) under assumption (5).

Let us set

$$I = \Big[i\colon \Big(\sum_k |a_{ik}|^t\Big) \leqslant \eta\Big].$$

Putting $F'$ instead of $F$ into (15) and (16), we obtain statement ($o$) for the first term on the right-hand side of (26). By (31), we see that under the assumptions of Theorem 6, 2°, the second term of (26) is $o\big(\mathrm{E}\varrho_{N_n}(t)\big)$. For the proof we consider the set

$$I = \Big[i\colon \Big(\sum_k |a_{ik}|^t\Big)^{1/t} \leqslant \eta\Big],$$

where $y^t F'(y)$ is sufficiently small for $y \geqslant 1/\eta$.

To estimate the third term, we observe that

$$\mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big] = 0 \quad \text{if } C\Big(\sum_k |a_{ik}|^t\Big)^{(1+2t)/3t} < \varepsilon$$

and, in the opposite case,

$$\sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik}\mathrm{E}Z_{ik}\Big| > \varepsilon\Big] p_{in} \leqslant C\sum_i\Big(\sum_k |a_{ik}|^t\Big)^2 p_{in} = o\big(\mathrm{E}\varrho_{N_n}(t)\big).$$

The last term in (26) is $o\big(\mathrm{E}\varrho_{N_n}(t)\big)$, for the exponent in (38) is greater than 2.

Thus the proof is completed.

Proof of Lemma 1. (A) $\Rightarrow$ (B). By ($a_2$) we have

$$M \geqslant \mathrm{E}|X|^t \geqslant \int_{-y}^{y} |x|^t d\mathrm{P}[X < x] + y^t \mathrm{P}[|X| \geqslant y]$$
$$\geqslant \int_{-y}^{y} |x|^t d\mathrm{P}[X < x] \to \mathrm{E}|X|^t \quad \text{as } y\to\infty.$$

Hence, by ($a_1$),

$$0 \leqslant y^t F(y) \leqslant y^t \mathrm{P}[|X| \geqslant y] \to 0 \quad \text{as } y\to\infty,$$

which proves ($b_1$). Assertion ($b_2$) follows from

$$M \geqslant \int_{-\infty}^{0} |y|^t d\mathrm{P}[X < y] + \int_{0}^{\infty} y^t d\mathrm{P}[X < y]$$

$$= \int_{0}^{\infty} (1 - P[X < y] + \mathrm{P}[X < -y])\, dy^t \geqslant \int_{0}^{\infty} F(y)\, dy^t = \int_{0}^{\infty} y^t |dF(y)|.$$

(B) $\Rightarrow$ (A). Let

$$G(y) = \begin{cases} 1 & \text{if } y = 0, \\ F(N) & \text{if } N < y \leqslant N+1. \end{cases}$$

We observe that $1 - G(y)$ is the distribution function of a random variable $X$. Then, for every $k \in N$,

$$\mathrm{P}[|X_k| \geqslant y] \leqslant F(y) \leqslant 1 - G(y) = \mathrm{P}[X \geqslant y]$$

and

$$\mathrm{E}|X|^t = \int_{0}^{\infty} y^t |dG(y)| = \sum_{k=1}^{\infty} k^t [F(k-1) - F(k)]$$

$$\leqslant 1 + 2^t \sum_{k=2}^{\infty} (k-1)^t [F(k-1) - F(k)] \leqslant 1 + 2^t M < \infty.$$

Thus we have proved Lemma 1.

Proof of Theorem 7. Let us estimate the right-hand side of (19). The first term can be bounded as (15) and (16) putting only $F'$ instead of $F$.

To estimate the second term let us observe that by (20) and Markov's inequality we have

$$\sum_i \mathrm{P}\Big[\Big|\sum_k a_{ik} \mathrm{E} Z_{ik}\Big| > \varepsilon\Big]\, p_{in} \leqslant C \sum_i \sum_k |a_{ik}| \int_{1/|a_{ik}|}^{\infty} y\, |dF'(y)|\, p_{in}. \tag{39}$$

By the assumption of Theorem 7 we see that

$$\int_{T}^{\infty} y\, |dF'(y)| \to 0 \quad \text{as } T \to \infty.$$

Therefore, using the method of the previous proofs, we see that (39) is $o(\mathrm{E}\varrho_{N_n}(t))$.

The last term of (19) can be estimated similarly as (24) and (25). This completes the proof.

Theorem 7′ follows from Theorem 7 and Lemma 1.

**3. Concluding remarks.**

1. One can observe that our theorems yield, among others, results contained in [1]-[3] and [5]-[8]. To see that it is enough to put $N_n = n$ a. s.

In this particular case, Theorem 4 is stronger than the corresponding result of [3]. Theorem 3 did not appear earlier.

2. Let $a_{ik}$ be such that (4) holds and

$$\sum_k |a_{ik}| \leqslant C.$$

For such real numbers, by Theorem 7′, $S_{N_n} \to 0$ in probability as $n \to \infty$. This fact extends the result of Rohatgi [7].

3. Let

$$a_{ik} = \begin{cases} 1/i^r & \text{for } 1 \leqslant k \leqslant i, \\ 0 & \text{for } k > i, \end{cases} \tag{40}$$

where $r > 1/t$. In this case,

$$\mathrm{E} \sum_k |a_{N_n k}|^t = \sum_i i^{1-rt} p_{in} = \mathrm{E} N_n^{1-rt},$$

and under the assumptions of Theorem 1, 1°, and Theorem 2, 1°, we have

$$\mathrm{P}\Big[\Big|\sum_{k=1}^{N_n} X_k\Big| > \varepsilon N_n^r\Big] = O(\mathrm{E} N_n^{1-rt}).$$

Theorem 3, 1°, gives

$$\mathrm{P}\Big[\Big|\sum_{k=1}^{N_n}\Big(X_k - \int_{-N_n}^{N_n} x\,dF_k(x)\Big)\Big| > \varepsilon N_n^r\Big] = O(\mathrm{E} N_n^{1-r})$$

and by Theorem 4, 1°,

$$\mathrm{P}\Big[\Big|\sum_{k=1}^{N_n}(X_k - \mathrm{E}X_k)\Big| > \varepsilon N_n^r\Big] = O(\mathrm{E} N_n^{1-rt}).$$

To obtain similar results for $t \geqslant 2$ we give the following

LEMMA 2. *If $a_i > 0$, $b_i > 0$, $p_i \geqslant 0$ for $i = 1, 2, \ldots$, and*

$$\sum_{i=1}^{\infty} a_i p_i < \infty, \quad \sum_{i=1}^{\infty} b_i p_i < \infty, \quad \text{and} \quad \sum_{i=1}^{\infty} p_i = 1,$$

*then*

$$\frac{\sum_{i=1}^{\infty} a_i p_i}{\sum_{i=1}^{\infty} b_i p_i} \leqslant \sup_i \frac{a_i}{b_i}.$$

From Lemma 2 we conclude that assumption (9) is satisfied for $a_{ik}$ given by (40) and for $0 < \lambda < 1$. Similarly, if

$$r > \frac{1}{2} \quad \text{and} \quad \alpha > \frac{rt-1}{2r-1},$$

then (10) is valid. Thus, if $t \geqslant 2$, then

$$\mathrm{P}\left[\left|\sum_{k=1}^{N_n}(X_k - \mathrm{E}X_k)\right| > \varepsilon N_n^r\right] = O(\mathrm{E}N_n^{1-rt}), \quad r > \frac{1}{2}.$$

To characterize convergence in probability to zero of sums with random indices we need

LEMMA 3. *If $N_n \to \infty$ in probability as $n \to \infty$, then, for an arbitrary $s < 0$, $\mathrm{E}N_n^s \to 0$ as $n \to \infty$.*

By Lemma 3, (4) is satisfied for

$$a_{ik} = \begin{cases} \left(\dfrac{1}{i}\right)^{1/t} & \text{for } 1 \leqslant k \leqslant i, \\ 0 & \text{for } k > i. \end{cases}$$

Hence, under the assumptions of Theorems 1, 3, and 4, if $N_n \to \infty$ in probability, then, for $0 < t < 1$, $t = 1$, and $1 < t < 2$, we get, respectively,

$$\mathrm{P}\left[\left|\sum_{k=1}^{N_n} X_k\right| > \varepsilon N_n^{1/t}\right] \to 0 \quad \text{as } n \to \infty,$$

$$\mathrm{P}\left[\left|\sum_{k=1}^{N_n}\left(X_k - \int_{-N_n}^{N_n} x\,dF_k(x)\right)\right| > \varepsilon N_n\right] \to 0 \quad \text{as } n \to \infty,$$

and

$$\mathrm{P}\left[\left|\sum_{k=1}^{N_n}(X_k - \mathrm{E}X_k)\right| > \varepsilon N_n^{1/t}\right] \to 0 \quad \text{as } n \to \infty.$$

4. It is known that if $Y_n \to 0$ in probability as $n \to \infty$, $N_n, Y_1, Y_2, \ldots$ are independent for every $n \geqslant 1$ and, moreover, $N_n \to \infty$ in probability, then $Y_{N_n} \to 0$ in probability as $n \to \infty$.

Theorems obtained in this paper give the conditions under which $Y_{N_n} \to 0$ in probability as $n \to \infty$ without the assumption that $Y_n \to 0$ in probability as $n \to \infty$. Furthermore, they furnish information about rates of convergence in probability.

Example. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent identically distributed random variables such that $\mathrm{E}|X_1| < \infty$ and $\mathrm{E}X_1 = 0$. Suppose that, for $p = 1, 2, \ldots$,

$$a_{ik} = \begin{cases} 1/i & \text{for } 1 \leqslant k \leqslant i,\ i = 2p, \\ 0 & \text{for } k > i,\ i = 2p, \\ 1 & \text{for } k = 1,\ i = 2p-1, \\ 0 & \text{for } k > 1,\ i = 2p-1. \end{cases}$$

One can observe that

$$\sup_k |a_{ik}| \nrightarrow 0 \quad \text{as } i \to \infty$$

and we cannot use Theorem 1 of [7] to assert that

$$S_i = \sum_k a_{ik} X_k \to 0 \text{ in probability} \quad \text{as } i \to \infty.$$

Thus we do not know if $S_i \to 0$ in probability as $i \to \infty$.

Now let $\{N_n, n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that, for $p = 1, 2, \ldots$,

$$\mathrm{P}[N_n = 1] = \frac{1}{n},$$

$$\mathrm{P}[N_n = i] = \begin{cases} \dfrac{n^{p-1} - (1/n)^{p-1}}{(p-1)!} \dfrac{1 - 1/n}{e^n - 1} & \text{for } i = 2p, \\ \dfrac{(1/n)^{p-1}}{(p-1)!} \dfrac{1 - 1/n}{e^n - 1} & \text{for } i = 2p+1. \end{cases}$$

We have

$$\mathrm{E} \sum_k |a_{N_n k}| = 1 \quad \text{and} \quad \mathrm{E} \sup_j |a_{N_n j}| \sum_k |a_{N_n k}| \to 0 \quad \text{as } n \to \infty.$$

Therefore, by Theorem 7′, $S_{N_n} \to 0$ in probability as $n \to \infty$.

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