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MAPS BETWEEN WEAK SOLENOIDAL SPACES

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1. Introduction. The purpose of this paper is to characterize induced maps between weak solenoidal spaces and to improve a theorem of Fort and McCord ([3], Theorem 1) concerning the approximation of maps between inverse limit spaces. We follow the notation of [2] for inverse limit systems. Given an inverse limit system (X, f) indexed by the directed set M, we have onto bonding maps f_a^β: $X_\beta \to X_a$ ($a \leqslant \beta$ in M) and projection maps f_a: $X_\infty \to X_a$ from the limit space X_∞ onto the factor spaces X_a. Let (Y, g) be a second inverse limit system (indexed by N). φ: $X_\infty \to Y_\infty$ is said to be an *induced map* if there is an order preserving function λ: $N \to M$ and a system of maps φ_n: $X_{\lambda(n)} \to Y_n$ (for each $n \in N$) with $\varphi_m f_{\lambda(m)}^{\lambda(n)} = g_m^n \varphi_n$ (if $m \leqslant n$) such that φ is defined by $g_m \varphi = \varphi_m f_{\lambda(m)}$.

The following question is stated in [3]: "Under what conditions on the systems (X, f) and (Y, g) can every map F: $X_\infty \to Y_\infty$ be approximated arbitrarily closely by induced maps?" This question is related to a problem posed by Mioduszewski [5]. Fort and McCord [3] give some sufficient conditions for such a map to be approximated by an induced map that is ε-homotopic to F. In Theorem 1 we weaken these conditions as well as show that the approximating map agrees with F on the inverse image (under F) of the "vertices" of Y_∞. In Theorem 3 we give necessary and sufficient conditions for the map F: $X_\infty \to Y_\infty$ itself to be an induced map, where (X, f) is an inverse limit system of compact, connected, Hausdorff spaces with all bonding maps onto and each projection f_n: $X_\infty \to X_n$ is an identification map, and where (Y, g) is a weak solenoidal sequence of polyhedra.

2. Notation and statements of results. A *weak solenoidal sequence* (*solenoidal sequence*) of polyhedra is an inverse limit sequence (Y, g) such that each factor space Y_n is a connected polyhedron and each bonding map g_m^n: $Y_n \to Y_m$ ($n \geqslant m$) is a covering map (regular covering map).

For a polyhedron Y with a given triangulation, we use the barycentric metric d, defined by

$$d(x, y) = \sum \{|x(v) - y(v)|: v \text{ a vertex of } Y\},$$

where $x(v)$ is the barycentric coordinate of $x \in Y$ with respect to v.

The limit space $Y_\infty = \lim(Y, g)$ of a weak solenoidal sequence of polyhedra is a compact metric space. The metric d_∞ may be given by

$$d_\infty(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n\big(g_n(x), g_n(y)\big)$$

(see [3]).

THEOREM 1. *Let (X, f) be an inverse limit system of compact, connected, Hausdorff spaces with all bonding maps onto. Let (Y, g) be a weak solenoidal sequence of polyhedra with Y_1 having any given triangulation. Then for any map $F: X_\infty \to Y_\infty$ and any $\varepsilon > 0$ there is an induced map $\varphi: X_\infty \to Y_\infty$ that agrees with F on the set $V = (f_1 F)^{-1}\{v: v$ a vertex of $Y_1\}$. Moreover, there is an ε-homotopy $h_\infty^t: X_\infty \to Y_\infty$ relative to V from φ to F. Hence $d_\infty(\varphi, F) < \varepsilon$.*

THEOREM 2. *Let (X, f) be an inverse limit sequence of polyhedra with all bonding maps onto and let (Y, g) be a weak solenoidal sequence of polyhedra with Y_1 having any given triangulation. Then for any map $F: X_\infty \to Y_\infty$ onto Y_∞ and any $\varepsilon > 0$ there is an induced map $\varphi: X_\infty \to Y_\infty$ onto Y_∞ that agrees with F on the set $V = (f_1 F)^{-1}\{v: v$ a vertex of $Y_1\}$. Moreover, there is an ε-homotopy $h_\infty^t: X_\infty \to Y_\infty$ relative to V from φ to F.*

A map $F: X_\infty \to Y_\infty$ is said to be *fiber preserving* if there is an integer n such that for each $x \in X_n$ there is a $y \in Y_1$ with $F\big(f_n^{-1}(x)\big) \subset g_1^{-1}(y)$.

THEOREM 3. *Let (X, f) be an inverse limit sequence of compact, connected, Hausdorff spaces with all bonding maps onto and each projection $f_n: X_\infty \to X_n$ an identification map. Let (Y, g) be a weak solenoidal sequence of polyhedra. Then a map $F: X_\infty \to Y_\infty$ is an induced map if and only if F is fiber preserving.*

3. Construction of approximating maps. Let (Y, g) be a weak solenoidal sequence of polyhedra with Y_1 having a given triangulation. We choose triangulations of each $Y_n (n \geqslant 2)$ so that all bonding maps g_m^n: $Y_n \to Y_m$ $(n \geqslant m)$ are simplicial maps. Let (X, g) be an inverse limit system of compact, connected, Hausdorff spaces with all bonding maps onto. The following lemma is a sharpening of the statement of Theorem X.11.9 of [2]:

LEMMA 1. *Let $h: X_\infty \to Y_n$ be a map (for some fixed n). Given any $\varepsilon > 0$ there is an index $\gamma \in M$ such that for every $\beta \geqslant \gamma$ there is a map*

$\psi: X_\beta \to Y_n$ and an ε-homotopy h_n^t: $\psi f_\beta \simeq h$ (rel V), where $V = h^{-1}\{x$: x a vertex of $Y_n\}$. Hence $d(\psi f_\beta, h) < \varepsilon$.

Proof. The proof given in [2] suffices if we observe that $\psi f_\beta | V = h|V$. This follows quickly, for if b is a vertex of Y_n, then b belongs to a unique open star neighborhood with respect to a subdivision Y_n' of Y_n, namely $\mathrm{st}(b, Y_n')$ (Y_n' is a subdivision of sufficiently fine mesh, depending on ε). Let τ be the covering of Y_n' by open star neighborhoods, let $\alpha = h^{-1}\tau$, and let δ be any open covering of X_β such that $f_\beta^{-1}(\delta)$ refines α. Consider a point $x \in h^{-1}(b)$ and any open set $U \in \delta$ containing $f_\beta(x)$. Then $f_\beta^{-1}(U) \subset h^{-1}(\mathrm{st}(b, Y_n'))$, and the latter is the only member of α containing $f_\beta^{-1}(U)$. Hence it follows from the proof of X.11.9 in [2] that $\psi f_\beta(x) = h(x)$. Thus ψf_β and h agree on V. Finally, the homotopy between ψf_β and h is constant on the set of points in X_∞ for which ψf_β and h agree.

The next lemma is standard (see, for example, [3]).

LEMMA 2. *Let A be any connected space. Suppose there exist an $\varepsilon > 0$, $m < n$, maps φ, ψ: $A \to Y_m$, and an ε-homotopy h_m^t: $\varphi \simeq \psi(\mathrm{rel}\, V)$ for some $V \subset A$. If there is a lifting $\tilde{\varphi}$ of φ with respect to the covering projection g_m^n: $Y_n \to Y_m$, then there is a lifting $\tilde{\psi}$ of ψ and an ε-homotopy h_n^t: $\tilde{\varphi} \simeq \tilde{\psi}$ (rel V) covering h_m^t. Furthermore, if φ, ψ, and $\tilde{\varphi}$ are onto maps, then $\tilde{\psi}$ is also an onto map, provided ε is sufficiently small.*

4. Proof of Theorem 1. Suppose $(X, f), (Y, g), F$: $X_\infty \to Y_\infty$, and $\varepsilon > 0$ are as in the statement of Theorem 1. We follow [3] to construct by recursion an induced map φ ε-approximating F. According to Lemma 1 there is an index $\gamma(1) \in M$, a map φ_1: $X_{\gamma(1)} \to Y_1$, and an ε-homotopy h_1^t: $\varphi_1 f_{\gamma(1)} \simeq g_1 F$ (rel V), where $V = (g_1F)^{-1}\{b$: b a vertex of $Y_1\}$. By the alternate use of Lemmas 1 and 2 we construct a sequence $\gamma(1) < \gamma(2) < \ldots$ of indices from M, a sequence of maps $\varphi_1, \varphi_2, \ldots,$ and a sequence of ε-homotopies $h_1^t, h_2^t, \ldots$ such that (for each n) φ_n: $X_{\gamma(n)} \to Y_n$, $\varphi_n f_{\gamma(n)}^{\gamma(n+1)} = g_n^{n+1}\varphi_{n+1}$, $g_n^{n+1}h_{n+1}^t = h_n^t$, and h_n^t: $\varphi_n f_{\gamma(n)} \simeq g_n F(\mathrm{rel}\, V)$. We have constructed an induced map φ: $X_\infty \to Y_\infty$ defined by the relation $g_n\varphi = \varphi_n f_{\gamma(n)}$ and an induced ε-homotopy h_∞^t: $\varphi \simeq F(\mathrm{rel}\, V)$ defined by the relation $g_n h_\infty^t = h_n^t$.

Remark. If (X, g) is an inverse limit system of polyhedra, the maps φ_n can be chosen to be simplicial (see Lemma 2 of [4]).

COROLLARY. *If $X_\infty = Y_\infty$ and F: $X_\infty \to Y_\infty$ is a homeomorphism, then φ is a homotopy equivalence.*

Proof. Apply Theorem 1 to F^{-1} and obtain $\psi \simeq F^{-1}$.

5. Proof of Theorem 2. We use the following result of Mardešić and Segal (Lemma 4 of [4]):

LEMMA 3. *Let X be a continuum, P_1 a polyhedron and f_1: $X \to P_1$ an ε_1-mapping onto P_1, and let $\delta > 0$ be an arbitrary positive number. Then* *there is an $\varepsilon_2 > 0$ such that, for any polyhedron P_2 and ε_2-mapping f_2: $X \to P_2$ onto P_2, there is a mapping π: $P_2 \to P_1$ onto P_1, such that the distance $d(f_1, \pi f_2) \leqslant \delta$.*

To prove Theorem 2 we observe that given any $\varepsilon_2 > 0$ there is an index n such that f_m: $Y_\infty \to Y_m$ is an ε-mapping, for every $m \geqslant n$. Now we follow the proof of Theorem 1, using Lemma 3 in place of Lemma 1 to obtain onto maps maps between the factor spaces (as in Lemma 1 we have $f_1|V = \pi f_2|V$ for $V = f_1^{-1}\{v\colon v \text{ a vertex of } P_1\}$). Lemma 2 assures us that the final sequence $\varphi_1, \varphi_2 \dots$ consists entirely of onto maps.

6. Proof of Theorem 3. Let (X, f) be an inverse limit sequence of compact, connected, Hausdorff spaces with each projection f_n: $X_\infty \to X_n$ an identification map (that is, the topology of X_n is the largest for which f_n is continuous), and let (Y, g) be a weak solenoidal sequence of polyhedra. It is clear that any induced map $X_\infty \to Y_\infty$ is fiber preserving, so we will only give an argument for the converse statement.

Suppose the map F: $X_\infty \to Y_\infty$ is fiber preserving. Then there is an integer $\gamma(1)$ such that $g_1 F$ is constant on each fiber $f_{\gamma(1)}^{-1}(x)$. Since $f_{\gamma(1)}$ is an identification map, $\varphi_1 = (g_1 F) f_{\gamma(1)}^{-1}$ is a continuous map ([1], Theorem VI. 3.2).

Now applying the same technique as used for the proof of Theorem 1, we define recursively a sequence of maps $\varphi_2, \varphi_3, \dots$ to get a diagram

$$\begin{array}{ccccccccc} X_{\gamma(1)} & \leftarrow & X_{\gamma(2)} & \leftarrow & X_{\gamma(3)} & \leftarrow & \dots & \leftarrow & X_\infty \\ \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & & & & \downarrow F \\ Y_1 & \leftarrow & Y_2 & \leftarrow & Y_3 & \leftarrow & \dots & \leftarrow & Y_\infty \end{array}$$

However, in this case we observe that the diagram is commutative, since at each step the maps $\varphi_n f_{\gamma(n)}$ and $g_n F$ are both liftings (with respect to g_{n-1}^n) that agree on at least one point of X_∞ and hence $\varphi_n f_{\gamma(n)} = g_n F$. Hence F is induced by the sequence $\varphi_1, \varphi_2, \dots$.

Remark. In light of Theorem 3, it is quite easy to construct maps (even homeomorphisms) which are not induced.

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