COLLOQUIUM MATHEMATICUM

VOL. XXXV 1976 FASC. 1

MEASURABLE SOLUTIONS OF QUADRATIC FUNCTIONAL EQUATIONS

BY

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1. Introduction. This note considers primarily measurable (Lebesgue on $\boldsymbol{R}^n$) solutions of certain systems of functional equations, the main result being that all measurable solutions are, in fact, $\mathscr{C}^\infty$ on $\boldsymbol{R}^n$. However, some of the results are valid for more general settings, and hence the problems will be posed in terms of functions with values in a field $\mathscr{F}$ and with domain $\mathscr{G}$, where $(\mathscr{G}, +)$ defines a commutative semigroup. This will be specialized later to the case $\mathscr{G} = \boldsymbol{R}^n$ and $\mathscr{F} = \boldsymbol{R}$.

The first problem considered is that of determining the $2N+1$ unknown functions $h, f_1, \ldots, f_N, g_1, \ldots, g_N$: $\mathscr{G} \to \mathscr{F}$ satisfying a functional equation of the form

(1) $$h(x+y) = A^{ij} f_i(x) g_j(y), \quad \text{where } i, j = 1, 2, \ldots, N,$$

and where, as through this note, the Einstein summation convention on repeated indices is used. The A^{ij} are assumed given, a system of N^2 elements in the field $\mathscr{F}$, subject to the condition

(2) $$\det(A^{ij}) \neq 0.$$

It is natural to assume

(3) $g_1, \ldots, g_N$ linearly independent,

(4) $f_1, \ldots, f_N$ linearly independent,

since otherwise certain terms in (1) could be combined so as to reduce N.

Methods developed by Vincze [10] have proved particularly useful in solving specific equations of form (1). Numerous references are given in [1] together with the following known result ([1], p. 199):

If $\mathscr{G} = \mathscr{F} = \boldsymbol{R}$ *and if, in addition to* (2), (3) *and* (4), *it is assumed that* h, g_i, f_i *have primitive functions* H, G_i, F_i, *then* h, g_i, f_i *are* $\mathscr{C}^\infty$ *on* $\boldsymbol{R}$.

The system of N simultaneous functional equations for N unknown functions g_i: $\mathscr{G} \to \mathscr{F}$, i.e.,

(5) $$g_k(x+y) = \Gamma_k^{ij} g_i(x) g_j(y), \quad \text{where } i, j, k = 1, 2, \ldots, N,$$

and where the Γ_k^{ij} denote a given system of N^3 elements from $\mathscr{F}$, has found numerous applications. To illustrate, Rota and Mullin [7] unified many results in combinatorics under the heading of "polynomials of binomial type". These are defined as polynomials $g_0(x), g_1(x), g_2(x), \ldots$ satisfying the recursive system of equations

$$(6) \qquad g_k(x+y) = \sum_{r=0}^{k} \binom{k}{r} g_{k-r}(x) g_r(y), \quad \binom{k}{r} = \frac{k!}{r!(k-r)!};$$

the polynomials so characterized in [7] include the Bell, Abel and Laguerre polynomials. System (6) has been used in [2] and [11] as the composition law for, and characterization of, certain Poisson distributions, while the general solution of (6), where $\mathscr{G}$ is a commutative semigroup and $\mathscr{F} = \boldsymbol{R}$ or $\mathscr{F} = \boldsymbol{C}$, was found by Aczél and Vranceanu [3] (see also Kuczma and Zajtz [6]).

Finally, the general equation (5) for continuous g_i and for $\mathscr{F}$, $\mathscr{G}$ either $\boldsymbol{R}$ or $\boldsymbol{C}$ was studied from the standpoint of hypercomplex numbers in [4], [8] and [9]; the Γ_k^{ij} are interpreted as the multiplication constants, relative to some basis, in an associative algebra with unit defined on the vector spaces $\boldsymbol{R}^N$ or $\boldsymbol{C}^N$.

2. Statement of results. The first theorem concerns the simultaneous system of equations (5) and relates various properties of the domain $(\mathscr{G}, +)$ with corresponding properties of the coefficients Γ_k^{ij}.

THEOREM 1. *Assume that* (5) *admits a solution* $g_i\colon \mathscr{G} \to \mathscr{F}$ *for a field* $\mathscr{F}$. *On the Cartesian product* $\mathscr{F}^N$ *define a binary operation*

$$\circ\colon \mathscr{F}^N \times \mathscr{F}^N \to \mathscr{F}^N$$

as follows: with $X = [X_1, \ldots, X_N]$ *and* $Y = [Y_1, \ldots, Y_N]$ *elements in* $\mathscr{F}^N$ *associate* $X \circ Y \in \mathscr{F}^N$ *expressed by*

$$(X \circ Y)_k = \Gamma_k^{ij} X_i Y_j,$$

so that $(\mathscr{F}^N, \circ, +)$ *forms an algebra over* $\mathscr{F}$. *Then the linear independence hypothesis* (3) *implies that if* $(\mathscr{G}, +)$ *is associative and/or commutative and/or with unit, then the algebra is associative and/or commutative and/or with unit, respectively.*

The second theorem shows that equation (1) is essentially equivalent to the system of equations (5); clearly, each equation in (5) is of type (1).

THEOREM 2. *Assume* $\mathscr{F}$ *a field, and* $(\mathscr{G}, +)$ *commutative and associative. If* $h, f_i, g_i\colon \mathscr{G} \to \mathscr{F}$ *satisfy* (1), *given* (2), (3) *and* (4), *then the* g_i (*and also the* f_i) *satisfy a system of equations of form* (5).

The remaining two theorems are specialized to the case $\mathscr{G} = \boldsymbol{R}^n$ and $\mathscr{F} = \boldsymbol{R}$, using Lebesgue measurability on $\boldsymbol{R}^n$ to deduce $\mathscr{C}^\infty$ on $\boldsymbol{R}^n$.

THEOREM 3. *If the g_i: $\boldsymbol{R}^n \to \boldsymbol{R}$ are (finite) Lebesgue measurable, linearly independent solutions of system* (5), *then the g_i are, in fact, $\mathscr{C}^\infty$ on $\boldsymbol{R}^n$.*

By Theorem 2, if h, f_i, g_i satisfy equation (1) subject to (2), (3) and (4), then both the f_i and the g_i satisfy a system of equations of form (5) to which Theorem 3 applies. The proof of Theorem 2 yields an even stronger result.

THEOREM 4. *Assume h, f_i, g_i: $\boldsymbol{R}^n \to \boldsymbol{R}$ satisfy* (1) *subject to* (2), (3) *and* (4). *Then each of the hypotheses*

(a) *h is measurable on $\boldsymbol{R}^n$,*

(b) *the g_i are measurable on $\boldsymbol{R}^n$ for $i = 1, \ldots, N$,*

(c) *the f_i are measurable on $\boldsymbol{R}^n$ for $i = 1, \ldots, N$,*

implies the other two and also implies $h, g_i, f_i \in \mathscr{C}^\infty$ on $\boldsymbol{R}^n$.

This generalizes the results in [1] and, in fact, using the methods given in [1], shows that the measurable solutions are of the "exponential polynomial" type (solutions of linear homogeneous differential equations with constant coefficients).

3. Proof of Theorem 1. If $(\mathscr{G}, +)$ is associative and/or commutative and/or with unit, say $0 \in \mathscr{G}$, then (5) implies correspondingly

$$(\Gamma_r^{ks}\Gamma_k^{ij} - \Gamma_r^{ik}\Gamma_k^{js}) g_i(x) g_j(y) g_s(z) = 0 \quad \text{for all } x, y, z \in \mathscr{G},$$

$$(\Gamma_k^{ij} - \Gamma_k^{ji}) g_i(x) g_j(y) = 0 \quad \text{for all } x, y \in \mathscr{G},$$

and

$$\Gamma_k^{ij} l_i g_j(x) = g_k(x) = \Gamma_k^{ij} g_i(x) l_j \quad \text{with } l_i = g_i(0),$$

which may be written as

$$(\Gamma_k^{ij} l_i - \delta_k^j) g_j(x) = 0 = (\Gamma_k^{ij} l_j - \delta_k^i) g_i(x) \quad \text{for all } x \in \mathscr{G}.$$

Here δ_k^i denotes the Kronecker delta ($I = (\delta_j^i)$ the identity matrix). For each fixed x, y the linear independence of $g_s(z)$, for each x that of $g_j(y)$, and that of $g_i(x)$ implies the vanishing of the coefficients. Hence

$$\Gamma_r^{ks}\Gamma_k^{ij} - \Gamma_r^{ik}\Gamma_k^{js} = 0, \quad \Gamma_k^{ij} - \Gamma_k^{ji} = 0, \quad \Gamma_k^{ij} l_i = \delta_k^j = \Gamma_k^{ji} l_i.$$

The theorem follows immediately upon multiplication of these equations by $X_i Y_j Z_s$, $X_i Y_j$ and X_j, respectively, for arbitrary $X, Y, Z \in \mathscr{F}^N$.

4. Proof of Theorem 2. It is well known [1] that a necessary and sufficient condition that $f_1, \ldots, f_N$: $\mathscr{G} \to \mathscr{F}$ be linearly independent, with $\mathscr{G}$ an abstract set and $\mathscr{F}$ a field, is that there exist elements $x_1, \ldots, x_N \in \mathscr{G}$ such that $\det(f_i(x_j)) \neq 0$; similarly for $g_1, \ldots, g_N$. Set $\gamma_{ik} = f_i(x_k)$ with the inverse matrix (γ^{ij}), $\gamma_{ij}\gamma^{jk} = \delta_i^k = \gamma^{kj}\gamma_{ji}$.

With this choice of x_k, by (2) equation (1) becomes

$$h(x_k + y) = A^{ij}\gamma_{ik} g_j(y) \quad \text{with } \det(A^{ij}\gamma_{ik}) \neq 0. \tag{7}$$

Hence

(8) $$g_j(y) = B_j^k h(x_k+y) \quad \text{with } \det(B_j^k) \neq 0,$$

and, in view of the complete symmetry of the problem, so also

(9) $$f_j(x) = C_j^k h(x+y_k) \quad \text{with } \det(C_j^k) \neq 0.$$

Note in passing that, for Theorem 4, equations (7), (8) and (9), together with the obvious

$$h(x+y_k) = A^{ij}\gamma_{jk}^* f_i(x), \qquad \gamma_{jk}^* = g_j(y_k),$$

show that (a), (b), (c) are equivalent assertions. By (8), together with commutativity and associativity in $(\mathscr{G}, +)$, we have

(10) $$g_j(x+y) = B_j^k h[(x_k+x)+y] = B_j^k A^{is} f_i(x_k+x) g_s(y)$$

while, by (9), we obtain

$$f_i(x_k+x) = C_i^r h[(x_k+y_r)+x] = C_i^r A^{pq} f_p(x_k+y_r) g_q(x)$$

which, when substituted in (10), yields (5). By symmetry, the $f_1, \ldots, f_N$ also satisfy a system of equations of type (5).

5. Proof of Theorem 3 ([1]). The proof will be based on Lebesgue's theorem on the almost everywhere differentiability of absolutely continuous functions of intervals (or balls) in $\boldsymbol{R}^n$. To be precise, the proof requires the following form of this theorem:

Let θ be open in $\boldsymbol{R}^n$, $0 < \mu(\theta) < \infty$ with μ Lebesgue measure. Suppose $f\colon \theta \to \boldsymbol{R}$ measurable and $|f| < K$ almost everywhere on θ. Then there exists a subset $\Omega \subset \theta$, $\mu(\Omega) = \mu(\theta)$, such that

(11) $$\lim_{\varrho\to 0} \frac{1}{\mu\{B_\varrho(x_0)\}} \int_{B_\varrho(x_0)} f\,dx = f(x_0) \quad \text{for all } x_0 \in \Omega,$$

where $B_\varrho(x_0)$ denotes the open ball about x_0 of radius $\varrho > 0$ (and sufficiently small to imply $B_\varrho(x_0) \subset \theta$).

In essence, with

$$F(B) = \int_B f\,dx,$$

assertion (11) becomes $DF = f$ almost everywhere on θ.

LEMMA. *Under the hypotheses of Theorem 3 there exists an open $\theta \subseteq \boldsymbol{R}^n$ on which all $|g_k|$ are bounded, say $|g_k| < K$ on θ for all $k = 1, 2, \ldots, N$.*

([1]) The author is indebted to a referee for many helpful suggestions and, in particular, for considerable simplification in this proof.

Proof. For arbitrary $U \subset \boldsymbol{R}^n$, $\mu(U) > 0$, write

$$S_k = \{x \mid x \in U,\ k \leqslant |g_1(x)| < k+1\}.$$

All S_k are measurable, disjoint and

$$\bigcup_{k=0}^{\infty} S_k = U.$$

Hence

$$\mu(U) = \sum_{k=0}^{\infty} \mu(S_k) > 0$$

implying that some S_{k_1} has positive measure with $|g_1| < k_1+1$ on $S_{k_1} \subset U$. Now replace U by S_{k_1}, and g_1 by g_2 to obtain $S_{k_2} \subset S_{k_1}$ for which $|g_2| < k_2+1$, $\mu(S_{k_2}) > 0$, while still maintaining $|g_1| < k_1+1$ on S_{k_2}. Repetition of this process yields a set $S \subset \boldsymbol{R}^n$ with $\mu(S) > 0$ on which all $|g_k|$ are bounded, say by K^*. But since the g_k satisfy (5), we have

$$|g_k| \leqslant |\Gamma_k^{ij}|_{\max} N^2 (K^*)^2 = K \quad \text{on } S+S.$$

But, by Steinhaus type theorems (see, for example, [5]), $S+S \supset \theta$ open.

To complete the proof, choose θ according to the Lemma. For $f = g_k$ let Ω_k be the set on which (11) holds. Since $\mu(\Omega_k) = \mu(\theta)$ for each $k = 1, \ldots, N$, so also

$$\mu\Big(\bigcap_{k=1}^{N} \Omega_k\Big) = \mu(\theta) > 0.$$

Hence there exists an $x_0 \in \bigcap_{k=1}^{N} \Omega_k$ for which (11) holds for all $f = g_k$, $k = 1, \ldots, N$. The system of equations (5) implies that

$$\int_{B_\varrho(x_0)} g_k(x+y)\,dx = \Gamma_k^{ij} \int_{B_\varrho(x_0)} g_i(x)\,dx\, g_j(y)$$

and, by a change of variable, this becomes, when divided by $\mu\{B_\varrho(x_0)\}$,

$$\frac{1}{\mu\{B_\varrho(x_0)\}} \int_{y+B_\varrho(x_0)} g_k(x)\,dx = A_k^j(\varrho) g_j(y), \tag{12}$$

where

$$A_k^j(\varrho) = \frac{1}{\mu\{B_\varrho(x_0)\}} \Gamma_k^{ij} \int_{B_\varrho(x_0)} g_i(x)\,dx.$$

But $A^j_k(\varrho)$ is continuous in $\varrho > 0$ and, by (11),

$$\lim_{\varrho\to 0} A^j_k(\varrho) = \Gamma^{ij}_k g_i(x_0)$$

(since $x_0 \in \Omega_k$ for each $k = 1, \ldots, N$) which can be taken as a definition of $A^j_k(0)$ with continuity for $\varrho \geqslant 0$. But, by (5),

$$g_k(x_0+y) = A^j_k(0) g_j(y),$$

and hence $\det(A^j_k(0)) \neq 0$, since otherwise there would exist non-trivial $\lambda^1, \ldots, \lambda^N$ for which $\lambda^k g_k(x_0+y) = 0$ for all y, contradicting linear independence. By continuity it follows that, for some ϱ_0,

$$\det(A^j_k(\varrho)) \neq 0 \quad \text{for all } 0 \leqslant \varrho < \varrho_0,$$

and (12) may be solved in the form

$$g_j(y) = B^k_j(\varrho) \int_{y+B_\varrho(x_0)} g_k(x)\,dx \quad \text{for arbitrary } \varrho \in (0, \varrho_0),$$

ifrom which follows the continuity, and hence $\mathscr{C}^\infty$, of all g_j.

6. Proof of Theorem 4. The equivalence of (a), (b) and (c) was shown n Section 4. By Theorem 2, the $f_1, \ldots, f_N$ and also the $g_1, \ldots, g_N$ satisfy a system of type (5) while, by Theorem 3, the f_i and g_i are then $\mathscr{C}^\infty$-functions, and so also is $h \in \mathscr{C}^\infty$ by (7).

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Reçu par la Rédaction le 19. 8. 1974;
en version modifiée le 5. 3. 1975