COLLOQUIUM MATHEMATICUM

VOL. XXXVIII 1977 FASC. 1

CLOSURE, INTERIOR, AND UNION IN FINITE TOPOLOGICAL SPACES

BY

LOUISE E. MOSER (HAYWARD, CALIFORNIA)

1. Introduction. Kuratowski [3] has shown that at most 14 distinct sets can be constructed from a subset A of a topological space X by application, in any order, of the closure, interior, and complement operators. In [2] Herda and Metzler proved that if 14 distinct sets can be obtained from A in this manner, then the cardinality of X is greater than or equal to 7. They showed further that there exist a topological space X of cardinality greater than or equal to 7 and a subset A of X from which exactly 14 sets can be obtained. Anusiak and Shum proved in [1] that if A is a subset of a topological space X of cardinality $n \leqslant 7$, then at most $2n$ sets can be constructed from A by closure, interior, and complementation.

A variation on Kuratowski's problem was posed recently by Smith [4]. Smith's problem was to show that if union replaces complement, then at most 13 distinct sets can be constructed from A. If $X = R^1$ with the usual topology, then

$$A = Q(0,1)\cup\left\{2-\frac{1}{n}\right\}_{n=1}^{\infty}\cup(3,4)-\left\{4-\frac{1}{n}\right\}_{n=1}^{\infty},$$

where $Q(0,1)$ is the set of rationals in the interval $(0,1)$, is a subset of R^1 from which 13 sets can be constructed.

In this paper, if A is a subset of a topological space X, we call ***associates*** of A the sets obtainable from A by application of the closure, interior, and union operators. We show that if a topological space X has a subset A with 13 distinct associates, then the cardinality of X is greater than or equal to 9. Further, we determine the maximum number of distinct associates of A for an arbitrary subset A of X if the cardinality of X is less than 9.

We note that if X is a finite T_1-space (points are closed), then every subset of X is closed as well as open; hence the number of associates of

A is equal to 1. If X is a finite T_0-space (given two distinct points, there exists a neighborhood of one not containing the other) and A is a subset of X, we show that the number of distinct associates of A is less than or equal to 7. We prove that if a subset A of X has 7 distinct associates, then the cardinality of X is greater than or equal to 6. If the cardinality of X is less than 6, we determine the maximum number of distinct associates of A.

In the following, we let $a(A)$ denote the number of distinct associates of A, and if X is of cardinality n, we let $a^*(n)$ denote the maximum number of associates of A for an arbitrary subset A of X. As usual, A^-, A°, and A' denote the closure, interior, and complement of A, respectively.

2. In this section we show that the cardinality of X is greater than or equal to 9 if X contains a subset A with 13 distinct associates. First, we give in the following diagram the possible distinct associates of A. The diagram is directed upward by set inclusion.

$$\begin{tikzcd}
& A^- \arrow[d, dash] & \\
& A^{-\circ-} \arrow[d, dash] & \\
& A^{\circ-}\cup A^{-\circ} \arrow[dl, dash] \arrow[dr, dash] & \\
A^{\circ-} \arrow[dr, dash] & & A^{-\circ} \arrow[dl, dash] \\
& A^{\circ-\circ} \arrow[d, dash] & \\
& A^\circ &
\end{tikzcd}$$

Fig. 1a

$$\begin{tikzcd}
& A^- = A^- \cup A \arrow[d, dash] & \\
& A^{-\circ-} \cup A \arrow[d, dash] & \\
& A^{\circ-}\cup A^{-\circ} \cup A \arrow[dl, dash] \arrow[dr, dash] & \\
A^{\circ-} \cup A \arrow[dr, dash] & & A^{-\circ} \cup A \arrow[dl, dash] \\
& A^{\circ-\circ} \cup A \arrow[d, dash] & \\
& A = A^\circ \cup A &
\end{tikzcd}$$

Fig. 1b

LEMMA 1. *If A is a subset of a topological space X and if $x \in A^{\circ-\prime} \cap A^{-\circ}$, then $\{x\}$ is not open.*

Proof. Assume that $x \in A^{\circ-\prime} \cap A^{-\circ}$ and $\{x\}$ is open. If $x \in A$, then $x \in A^\circ$ since $\{x\}$ is open; hence $x \in A^{\circ-}$ which is a contradiction. If $x \notin A$, then $x \notin A^-$ since $\{x\}$ is open; hence $x \notin A^{-\circ}$ which is a contradiction. (Or see [2].)

LEMMA 2. *If any one of the equalities*

(1) $A^{\circ-\circ} = A^{-\circ}$,

(2) $A^{\circ-} \cup A^{-\circ} = A^{\circ-}$,

(3) $A^{-\circ-} = A^{\circ-}$,

(4) $A^{\circ-\prime} \cap A^{-\circ} = \emptyset$

occurs, then all four occur and $a(A) \leqslant 7$.

Proof. If $A^{\circ-\circ} = A^{-\circ}$, then

$$A^{-\circ} \subseteq A^{\circ-} \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} = A^{\circ-}.$$

If $A^{\circ-} \cup A^{-\circ} = A^{\circ-}$, then $A^{-\circ} \subseteq A^{\circ-}$, so $A^{-\circ-} \subseteq A^{\circ-}$, but $A^{\circ-} \subseteq A^{-\circ-}$, hence $A^{-\circ-} = A^{\circ-}$. If $A^{-\circ-} = A^{\circ-}$, then

$$A^{-\circ} \subseteq A^{\circ-} \quad \text{and} \quad A^{\circ-\prime} \cap A^{-\circ} = \emptyset.$$

If $A^{\circ-\prime} \cap A^{-\circ} = \emptyset$, then $A^{-\circ} \subseteq A^{\circ-}$, so $A^{-\circ} \subseteq A^{\circ-\circ}$, but $A^{\circ-\circ} \subseteq A^{-\circ}$, hence $A^{\circ-\circ} = A^{-\circ}$. It follows that $a(A) \leqslant 7$.

Lemma 3. *If $A^{\circ} = \emptyset$ or if $A^{-} = X$, then $a(A) \leqslant 7$.*

Proof. If $A^{\circ} = \emptyset$, then

$$A^{\circ} = A^{\circ-\circ} = A^{\circ-} = \emptyset \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} = A^{-\circ},$$

so

$$A^{\circ} \cup A = A^{\circ-\circ} \cup A = A^{\circ-} \cup A = A \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} \cup A = A^{-\circ} \cup A,$$

thus $a(A) \leqslant 7$. If $A^{-} = X$, then

$$A^{-} = A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = X,$$

so

$$A^{-} \cup A = A^{-\circ} \cup A = A^{\circ-} \cup A^{-\circ} \cup A = A^{-\circ-} \cup A = X \quad \text{and} \quad a(A) \leqslant 7.$$

Lemma 4. *If $A^{\circ} = A$ or if $A^{-} = A$, then $a(A) \leqslant 3$.*

Proof. If $A^{\circ} = A$, then

$$A^{-} = A^{\circ-} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-},$$

and also $A^{\circ-\circ} = A^{-\circ}$ by Lemma 2. Further, no new associates are obtained by taking unions, thus $a(A) \leqslant 3$. If $A = A^{-}$, then $A^{-\circ} = A^{\circ} = A^{\circ-\circ}$ and, therefore,

$$A^{\circ-} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-}$$

by Lemma 2. Further, taking unions just gives A^{-} again, thus $a(A) \leqslant 3$.

Theorem 1. *If a topological space X contains a subset A which has 13 distinct associates, then the cardinality of X is greater than or equal to 9.*

Proof. Let A be a subset of X with 13 distinct associates; then $A^{\circ-}$ and $A^{-\circ-}$ are distinct. Hence $A^{\circ-\prime} \cap A^{-\circ} \neq \emptyset$ by Lemma 2; in fact, $A^{\circ-\prime} \cap A^{-\circ}$ contains at least two points by Lemma 1. Let $A^{\circ} = \{x_1, \ldots, x_n\}$.

A construction of 13 distinct associates from A° by adding a minimum number of points is shown in Fig. 2a.

$$\begin{tikzcd}[column sep=tiny]
& A^{-} = \{x_1, \ldots, x_{n+5}, x_{n+6}, x_{n+7}\} \arrow[d, dash] & \\
& A^{-\circ-} = \{x_1, \ldots, x_{n+4}, x_{n+5}\} \arrow[d, dash] & \\
& A^{\circ-} \cup A^{-\circ} = \{x_1, \ldots, x_{n+4}\} \arrow[dl, dash] \arrow[dr, dash] & \\
A^{\circ-} = \{x_1, \ldots, x_{n+1}, x_{n+2}\} \arrow[dr, dash] & & A^{-\circ} = \{x_1, \ldots, x_{n+1}, x_{n+3}, x_{n+4}\} \arrow[dl, dash] \\
& A^{\circ-\circ} = \{x_1, \ldots, x_n, x_{n+1}\} \arrow[d, dash] & \\
& A^{\circ} = \{x_1, \ldots, x_n\} &
\end{tikzcd}$$

Fig. 2a

$$\begin{tikzcd}[column sep=tiny]
& A^{-} = A^{-} \cup A \arrow[d, dash] & \\
& A^{-\circ-} \cup A = \{x_1, \ldots, x_{n+4}, x_{n+5}, x_{n+7}\} \arrow[d, dash] & \\
& A^{\circ-} \cup A^{-\circ} \cup A = \{x_1, \ldots, x_{n+4}, x_{n+7}\} \arrow[dl, dash] \arrow[dr, dash] & \\
A^{\circ-} \cup A = \{x_1, \ldots, x_{n+1}, x_{n+2}, x_{n+4}, x_{n+7}\} \arrow[dr, dash] & & A^{-\circ} \cup A = \{x_1, \ldots, x_{n+1}, x_{n+3}, x_{n+4}, x_{n+7}\} \arrow[dl, dash] \\
& A^{\circ-\circ} \cup A = \{x_1, \ldots, x_n, x_{n+1}, x_{n+4}, x_{n+7}\} \arrow[d, dash] & \\
& A = A^{\circ} \cup A = \{x_1, \ldots, x_n, x_{n+4}, x_{n+7}\} &
\end{tikzcd}$$

Fig. 2b

It follows from Fig. 2b that $x_{n+4} \in A$, since $x_{n+3}, x_{n+4} \in A^-$ and $A^{\circ-\prime} \cap A^{-\circ} = \{x_{n+3}, x_{n+4}\}$ is open; also $x_{n+7} \in A$, otherwise,

$$A^{-\circ} \cup A = A^{-\circ}, \qquad A^{\circ-} \cup A^{-\circ} \cup A = A^{\circ-} \cup A^{-\circ},$$

and

$$A^{-\circ-} \cup A = A^{-\circ-}.$$

Since A has 13 associates, $A^\circ \neq \emptyset$ and $A^- \neq X$ by Lemma 3. Hence $n \geqslant 1$, and $n+7$ is less than the cardinality of X. Therefore, the cardinality of X is greater than or equal to 9.

Further, if X is of cardinality $n \geqslant 9$, then there are a subset A of X and a topology τ on X such that $a^*(n) = a(A) = 13$. For let

$$A = \{x_1, x_5, x_8, x_{10}, x_{11}, x_{12}, \ldots, x_n\}$$

and let

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_4, x_5, x_6, x_7, x_8, x_9\}, \{x_1, x_2, x_4, x_5\}, \{x_7, x_8, x_9\}, \{x_9\}\}.$$

Then $\mathscr{B}$ is a basis for the weakest topology τ on X such that A has 13 distinct associates. This example, of course, is not unique.

3. In this section we determine the maximum number of associates of A if A is an arbitrary subset of X and the cardinality of X is less than 9.

THEOREM 2. *If X is a topological space of cardinality $n < 9$, then $a^*(n)$, the maximum number of associates of A for an arbitrary subset A of X, is given in the following table:*

n	1	2	3	4	5	6	7	8
$a^*(n)$	1	3	3	5	7	8	10	12

Proof. First we note that if $A = \emptyset$ or if $A = X$, then $a(A) = 1$. In particular, if $n = 1$, then $a^*(1) = 1$. So we assume in the following that $A \neq \emptyset$ and $A \neq X$. Further, if $A^\circ = A$ or if $A^- = A$, then $a(A) \leqslant 3$ by Lemma 4. So we also assume that $A^\circ \neq A$ and $A^- \neq A$ in the sequel.

If $n = 2$, then it remains to consider the case $A = \{x_1\}$, $A^\circ = \emptyset$, and $A^- = X$. In this case, $a(A) = 3$ and the trivial topology on X gives the maximum number of 3 associates.

If $n = 3$, then first consider $A = \{x_1\}$. If $A^\circ = \emptyset$ and if $A^- = \{x_1, x_2\}$ or if $A^- = X$, then $a(A) = 3$. Next consider $A = \{x_1, x_2\}$. If $A^- = X$ and if $A^\circ = \emptyset$ or if A° is a singleton, then $a(A) = 3$. For $n = 3$ it is not difficult to check that there are 5 topologies on X which give the maximum number of 3 associates for a subset A of X, one of which is the trivial topology with $A = \{x_1\}$ or $A = \{x_1, x_2\}$.

For $n = 4$ we show that $a^*(4) = 5$. First, let $A = \{x_1\}$. If $A^- = \{x_1, x_2\}$, then A^- has just 4 subsets, so $a(A) \leqslant 4$.

If $A^- = \{x_1, x_2, x_3\}$ and if $A^\circ = \emptyset$, then

$$A^\circ = A^{\circ-\circ} = A^{\circ-} = \emptyset \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} = A^{-\circ}.$$

If $A^{-\circ} = \emptyset$, then

$$A^{-\circ-} = A^{\circ-} \cup A^{-\circ} = \emptyset \quad \text{and} \quad a(A) = 3.$$

If $A^{-\circ} \neq \emptyset$, then, since $x_2, x_3 \in A^-$ and $A = \{x_1\}$, we have $x_1 \in A^{-\circ}$ and $a(A) \leqslant 5$.

If $A^- = \{x_1, x_2, x_3, x_4\} = X$ and if $A^\circ = \emptyset$, then

$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = A^- = X, \quad A^\circ = A^{\circ-\circ} = A^{\circ-} = \emptyset$$

and

$$a(A) = 3.$$

Next, let $A = \{x_1, x_2\}$. If $A^- = \{x_1, x_2, x_3\}$ and if $A^\circ = \{x_1\}$, then $\{x_1, x_3\}$ is the only other possible distinct associate of A in Fig. 1a, and $\alpha(A) \leqslant 4$. So suppose that $A^\circ = \emptyset$; then

$$A^\circ = A^{\circ-\circ} = A^{\circ-} = \emptyset \quad \text{and} \quad A^{\circ-} \cup A^{-\circ} = A^{-\circ}.$$

If $A^{-\circ} = \emptyset$, then $A^{-\circ-} = A^{\circ-} \cup A^{-\circ} = \emptyset$ and $\alpha(A) = 3$. If $A^{-\circ} \neq \emptyset$, then, since $x_3 \in A^-$ and $A = \{x_1, x_2\}$, we have

$$A^{-\circ} = \{x_1, x_3\} \text{ or } \{x_2, x_3\} \quad \text{and} \quad \alpha(A) = 4.$$

If $A^- = \{x_1, x_2, x_3, x_4\} = X$, then

$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = A^- = X.$$

If $A^\circ = \emptyset$, then $A^\circ = A^{\circ-\circ} = A^{\circ-} = \emptyset$ and $\alpha(A) = 3$. So assume that $A^\circ = \{x_1\}$; then $x_1 \in A^{\circ-}$ and $A^{\circ-}$ is not a 3-point set, since $A^{\circ-\prime} \cap A^{-\circ}$ is not a singleton by Lemma 1. If $A^{\circ-} = \{x_1\}$ or $A^{\circ-} = \{x_1, x_2\}$, then

$$A^{\circ-} \cup A = A^{\circ-\circ} \cup A = A^\circ \cup A = A \quad \text{and} \quad \alpha(A) = 3.$$

If $A^{\circ-} = \{x_1, x_3\}$, then

$$A^{\circ-} \cup A = \{x_1, x_2, x_3\}, \quad A^{\circ-\circ} \cup A = \{x_1, x_2, x_3\} \text{ or } \{x_1, x_2\}$$

and

$$\alpha(A) = 5.$$

If $A^{\circ-} = \{x_1, x_2, x_3, x_4\} = X$, then $A^{\circ-\circ} = X$ and $\alpha(A) = 3$.

Finally, let $A = \{x_1, x_2, x_3\}$ and assume that $A^- = X$; then

$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = A^- = X.$$

If $x_4 \in A^{\circ-}$, then $A^{\circ-} \cup A = X$; whereas if $x_4 \notin A^{\circ-}$, then $A^{\circ-} \cup A = A$. Similarly, if $x_4 \in A^{\circ-\circ}$, then $A^{\circ-\circ} \cup A = X$; whereas, if $x_4 \notin A^{\circ-\circ}$, then $A^{\circ-\circ} \cup A = A$. Hence $\alpha(A) \leqslant 5$.

If $A = \{x_1, x_2\}$ and $\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_2, x_4\}\}$, then $\mathscr{B}$ is a basis for a topology τ on X such that $\alpha(A) = \alpha^*(4) = 5$.

A different approach will be used in the remaining cases.

$n = 5$. We show that $\alpha^*(5) = 7$ by giving an example of a subset A of X and a topology τ on X such that $\alpha(A) = 7$. Let

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_1, x_3\}, \{x_2, x_5\}\}$$

be a basis for the topology τ on X and let $A = \{x_1, x_2\}$; then $\alpha(A) = 7$.

If $A^\circ = \emptyset$ or if $A^- = X$, then $\alpha(A) \leqslant 7$ by Lemma 3; so we will assume that $A^\circ \neq \emptyset$ and $A^- \neq X$. If $A^{\circ-\prime} \cap A^{-\circ} = \emptyset$, then $\alpha(A) \leqslant 7$ by Lemma 2; so we will assume that $A^{\circ-\prime} \cap A^{-\circ} \neq \emptyset$. We have already assumed that $A^\circ \neq A$ and $A^- \neq A$ since, otherwise, $\alpha(A) \leqslant 3$ by Lemma 4.

Given a subset A of X, at least 3 pairs of associates of A in Fig. 1a are equal since X is of cardinality $n = 5$. If more than 3 pairs of associates of A in Fig. 1a are equal, then $\alpha(A) \leqslant 5$. So it remains to consider the case in which exactly 3 pairs of associates are equal. In this case the cardinality of A° is equal to 1 and the cardinality of A^- is equal to 4, since exactly 3 pairs of associates of A are equal and the cardinality of X is equal to 5.

If $A^{\circ-\circ} = A^{\circ-}$, then

$$A^{\circ-} \subseteq A^{-\circ} \quad \text{and} \quad A^{-\circ} = A^{\circ-} \cup A^{-\circ}.$$

If, in addition, $A^\circ = A^{\circ-\circ}$, then

$$A^\circ = A^{\circ-\circ} = A^{\circ-} = \{x_1\}, \quad A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3\},$$
$$A^{-\circ-} = \{x_1, x_2, x_3, x_4\}, \quad \text{and} \quad A^- = \{x_1, \ldots, x_5\} = X,$$

but then

$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = A^- = X,$$

which is a contradiction. If instead $A^- = A^{-\circ-}$, then

$$A^\circ = \{x_1\}, \quad A^{\circ-\circ} = A^{\circ-} = \{x_1, x_2\},$$
$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\}, \quad \text{and} \quad A^{-\circ-} = A^- = X,$$

which is also a contradiction since more than 3 pairs of associates of A in Fig. 1a are equal.

If $A^\circ = A^{\circ-\circ}$, $A^{-\circ} = A^{\circ-} \cup A^{-\circ}$, and $A^{-\circ-} = A^-$, then

$$A^\circ = A^{\circ-\circ} = \{x_1\}, \quad A^{\circ-} = \{x_1, x_2\},$$
$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\}, \quad \text{and} \quad A^{-\circ-} = A^- = X,$$

which is again a contradiction.

$n = 6$. We show that $\alpha^*(6) = 8$ by first noting that if

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_3, x_4, x_5, x_6\}, \{x_1, x_3, x_4\}, \{x_6\}\}$$

and

$$A = \{x_1, x_3, x_5\},$$

then $\mathscr{B}$ is a basis for a topology τ on X such that $\alpha(A) = 8$. As in the case $n = 5$, we assume that

$$A^\circ \neq \emptyset, \quad A^- \neq X, \quad A^{\circ-\prime} \cap A^{-\circ} \neq \emptyset, \quad A^\circ \neq A, \quad \text{and} \quad A^- \neq A$$

Given a subset A of X, at least 2 pairs of associates of A in Fig. 1a are equal since X is of cardinality $n = 6$. If more than 2 pairs of associates in Fig. 1a are equal, then $\alpha(A) \leqslant 7$. So it remains to consider the case in which exactly 2 pairs of associates of A are equal. In this case the cardinality of A° is equal to 1 and the cardinality of A^- is equal to 5, since exactly 2 pairs of associates of A are equal and the cardinality of X is equal to 6.

If $A^{\circ-\circ} = A^{\circ-}$, then $A^{\circ-} \subseteq A^{-\circ}$ and $A^{-\circ} = A^{\circ-} \cup A^{-\circ}$. Hence
$$A^{\circ} = \{x_1\}, \quad A^{\circ-\circ} = A^{\circ-} = \{x_1, x_2\}, \quad A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\},$$
$$A^{-\circ-} = \{x_1, \ldots, x_5\}, \quad \text{and} \quad A^{-} = \{x_1, \ldots, x_6\} = X,$$
but then
$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-} = A^{-} = X,$$
and more than 2 pairs of associates of A in Fig. 1a are equal, which is a contradiction.

If $A^{-\circ} = A^{\circ-} \cup A^{-\circ}$ and $A^{-\circ-} = A^{-}$, then
$$A^{\circ} = \{x_1\}, \quad A^{\circ-\circ} = \{x_1, x_2\}, \quad A^{\circ-} = \{x_1, x_2, x_3\},$$
$$A^{-\circ} = \{x_1, \ldots, x_5\}, \quad \text{and} \quad A^{-\circ-} = A^{-} = \{x_1, \ldots, x_6\} = X,$$
which is a contradiction.

If $A^{-\circ} = A^{\circ-} \cup A^{-\circ}$ and $A^{\circ} = A^{\circ-\circ}$, then
$$A^{\circ} = A^{\circ-\circ} = \{x_1\}, \quad A^{\circ-} = \{x_1, x_2\},$$
$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\},$$
$$A^{-\circ-} = \{x_1, \ldots, x_5\}, \quad \text{and} \quad A^{-} = \{x_1, \ldots, x_6\} = X,$$
which is again a contradiction.

If $A^{\circ} = A^{\circ-\circ}$ and $A^{-\circ-} = A^{-}$, then
$$A^{\circ} = A^{\circ-\circ} = \{x_1\}, \quad A^{\circ-} = \{x_1, x_2\}, \quad A^{-\circ} = \{x_1, x_3, x_4\},$$
$$A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\}, \quad \text{and} \quad A^{-\circ-} = A^{-} = \{x_1, \ldots, x_5\}.$$

Hence, $x_1 \in A$ and x_3 or $x_4 \in A$ since $A^{\circ-\prime} \cap A^{-\circ} = \{x_3, x_4\}$ is open and $x_3, x_4 \in A^{-}$. If $x_5 \in A$, then
$$A^{\circ-} \cup A^{-\circ} \cup A = A \quad \text{and} \quad \alpha(A) \leqslant 8.$$

If $x_5 \notin A$, then
$$A^{\circ-} \cup A^{-\circ} \cup A = A^{\circ-} \cup A^{-\circ} \quad \text{and} \quad \alpha(A) \leqslant 8.$$

$n = 7$. We show that $\alpha^*(7) = 10$ by first noting that if
$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_4, x_5, x_6, x_7\}, \{x_1, x_2, x_4, x_5\}, \{x_6, x_7\}, \{x_7\}\}$$
and
$$A = \{x_1, x_4, x_6\},$$
then $\mathscr{B}$ is a basis for a topology τ on X such that $\alpha(A) = 10$. As in cases $n = 5$ and $n = 6$, we assume that
$$A^{\circ} \neq \emptyset, \quad A^{-} \neq X, \quad A^{\circ-\prime} \cap A^{-\circ} = \emptyset, \quad A^{\circ} \neq A, \quad \text{and} \quad A^{-} \neq A.$$

Given a subset A of X, at least 2 associates of A in Fig. 1a are equal since X is of cardinality $n = 7$. If more than 2 associates of A in Fig. 1a are equal, then $\alpha(A) \leqslant 9$. So we consider the remaining case in which exactly 2 associates are equal. Since the cardinality of X is equal to 7 and exactly 2 associates are equal, the cardinality of A° is equal to 1 and the cardinality of A^- is equal to 6.

If $A^{\circ-\circ} = A^\circ = \{x_1\}$, then

$$A^{\circ-} = \{x_1, x_2\}, \quad A^{-\circ} = \{x_1, x_3, x_4\}, \quad A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4\},$$
$$A^{-\circ-} = \{x_1, \ldots, x_5\}, \quad \text{and} \quad A^- = \{x_1, \ldots, x_6\}.$$

Hence $x_1 \in A$ since $x_1 \in A^\circ$; also $x_6 \in A$ since $x_6 \in A^-$ and $\{x_6, x_7\}$ is open. Thus $A^{\circ-\circ} \cup A = A$ and $A^{-\circ-} \cup A = A^-$, so $\alpha(A) \leqslant 10$.

If $A^{-\circ} = A^{\circ-} \cup A^{-\circ}$, then more than 2 associates of A in Fig. 1a are equal. For if $A^{\circ-\circ} \neq A^{-\circ}$, then $A^{-\circ-} = A^-$ since

$$A^\circ = \{x_1\}, \quad A^{\circ-\circ} = \{x_1, x_2\}, \quad A^{\circ-} = \{x_1, x_2, x_3\},$$
$$A^{-\circ} = A^{\circ-} \cup A^{-\circ} = \{x_1, x_2, x_3, x_4, x_5\}, \quad A^{-\circ-} = \{x_1, \ldots, x_6\} = A^-;$$

for otherwise,

$$A^- = X = A^{-\circ} = A^{\circ-} \cup A^{-\circ} = A^{-\circ-}.$$

If $A^{-\circ-} = A^-$, then

$$A^\circ = \{x_1\}, \quad A^{\circ-\circ} = \{x_1, x_2\}, \quad A^{\circ-} = \{x_1, x_2, x_3\},$$
$$A^{-\circ} = \{x_1, x_2, x_4, x_5\},$$
$$A^{\circ-} \cup A^{-\circ} = \{x_1, \ldots, x_5\}, \quad \text{and} \quad A^{-\circ-} = \{x_1, \ldots, x_6\} = A^-.$$

Hence, $x_1 \in A$ and x_4 or $x_5 \in A$ since $A^{\circ-\prime} \cap A^{-\circ} = \{x_4, x_5\}$ is open. Further, if $x_6 \in A$, then

$$A^{\circ-} \cup A^{-\circ} \cup A = A^-,$$

and if $x_6 \notin A$, then

$$A^{\circ-} \cup A^{-\circ} \cup A = A^{\circ-} \cup A^{-\circ}.$$

Hence $\alpha(A) \leqslant 10$.

$n = 8$. By Theorem 1 the maximum number of associates of A for an arbitrary subset A of X is less than or equal to 12. Let $A = \{x_1, x_5, x_7\}$ and let

$$\mathcal{B} = \big\{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_4, \ldots, x_8\}, \{x_1, x_2, x_4, x_5\}, \{x_7, x_8\}, \{x_8\}\big\};$$

then $\mathcal{B}$ is a basis for a topology τ on X such that $\alpha(A) = \alpha^*(8) = 12$. We remark that τ is the weakest topology on X such that $\alpha(A) = 8$ and that this example is not unique.

4. Finally, in this section we draw some conclusions about associates of A in a finite T_0-space X.

THEOREM 3. *If X is a finite T_0-space and A is a subset of X, then $\alpha(A)$, the maximum number of associates of A, is less than or equal to 7. If $\alpha(A) = 7$, then the cardinality of X is greater than or equal to 6.*

Proof. In a finite T_0-space, $A^{-\circ} = A^{\circ-\circ}$ and $A^{\circ-} = A^{-\circ-}$ (see Theorem 3 in [2]). Hence $A^{\circ-} \cup A^{-\circ} = A^{\circ-}$, and $\alpha(A) \leqslant 7$ by Lemma 2. If $\alpha(A) = 7$, then the cardinality of X is greater than or equal to 5 by Theorem 2.

If $\alpha(A) = 7$ and the cardinality of X is equal to 5, then we must have

$$A^\circ = \{x_1\}, \quad A^{-\circ} = \{x_1, x_2\},$$
$$A^{\circ-} = \{x_1, x_2, x_3\}, \quad \text{and} \quad A^- = \{x_1, x_2, x_3, x_4\}.$$

But if $x_4 \in A$, then $A^{\circ-} \cup A = A^-$; whereas if $x_4 \notin A$, then $A^{\circ-} \cup A = A^{\circ-}$. In either case $\alpha(A) \leqslant 6$, and we have a contradiction.

However, if the cardinality of X is equal to 6, then there exist a T_0-topology τ on X and a subset A of X such that $\alpha(A) = 7$. For let

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_4, x_5, x_6\}, \{x_5, x_6\}, \{x_6\}\},$$
$$A = \{x_1, x_5\};$$

then $\mathscr{B}$ is a basis for a T_0-topology τ on X such that A has 7 distinct associates.

THEOREM 4. *If X is a T_0-space of cardinality $n < 6$, then $\alpha^*(n)$, the maximum number of associates of A for an arbitrary subset A of X, is given in the following table:*

n	1	2	3	4	5
$\alpha^*(n)$	1	2	3	5	6

Proof. If $n = 1$, then $\tau = \{X, \emptyset\}$, which is vacuously a T_0-topology on X, and $\alpha^*(1) = 1$. If $n = 2$, then the trivial topology is not T_0 and the discrete topology gives $\alpha(A) = 1$ for all subsets A of X. The remaining topology $\tau = \{X, \emptyset, \{x_1\}\}$ is T_0 and gives $\alpha(A) = 2$ for $A = \{x_1\}$ or $A = \{x_2\}$. If $n = 3$, then $\alpha^*(3) \leqslant 3$ by Theorem 2; if

$$\tau = \{X, \emptyset, \{x_3\}, \{x_1, x_3\}\} \quad \text{and} \quad A = \{x_1\},$$

then τ is a T_0-topology on X such that A has 3 distinct associates. If $n = 4$, then $\alpha^*(4) \leqslant 5$ by Theorem 2; further, if

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_3, x_4\}\} \quad \text{and} \quad A = \{x_1, x_3\},$$

then $\mathscr{B}$ is a basis for a T_0-topology τ on X such that A has 5 distinct associates. If $n = 5$, then, by the proof of Theorem 3, $a^*(5) \leqslant 6$; further, if

$$\mathscr{B} = \{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_4, x_5\}, \{x_5\}\} \quad \text{and} \quad A = \{x_1, x_4\},$$

then $\mathscr{B}$ is a basis for a T_0-topology τ on X such that A has 6 distinct associates.

REFERENCES

[1] J. Anusiak and K. P. Shum, *Remarks on finite topological spaces*, Colloquium Mathematicum 23 (1971), p. 217-223.

[2] H. H. Herda and R. C. Metzler, *Closure and interior in finite topological spaces*, ibidem 15 (1966), p. 211-216.

[3] C. Kuratowski, *Sur l'opération $\bar{A}$ de l'analysis situs*, Fundamenta Mathematicae 3 (1922), p. 182-199.

[4] A. Smith, *Advanced problem 5996*, The American Mathematical Monthly 81 (1974), p. 1034.

Reçu par la Rédaction le 15. 8. 1975