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HOPF'S EXTENSION THEOREM IN THE THEORY OF SHAPE

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One of the classical theorems of the homotopy theory is the following (see [2], p. 147):

HOPF'S EXTENSION THEOREM. *Let $\hat{X}$ be a compactum with* $\dim \hat{X} \leqslant n+1$ *and let X be a closed subset of $\hat{X}$. In order a map $f: X \to S^n$ be extendable to a map $\hat{f}: \hat{X} \to S^n$ it is necessary and sufficient that the kernel of the homomorphism $f_*: H_n(X) \to H_n(S^n)$ induced by f contain the kernel of the homomorphism $j_*: H_n(X) \to H_n(\hat{X})$ induced by the inclusion map $j: X \to \hat{X}$.*

By $H_n(X)$ we mean here the n-th topological homology group of X (in the sense of Vietoris or of Čech) with coefficients being real numbers modulo 1.

The aim of this note is to transfer this theorem into the theory of shape (concerning this notion see [1]) in the following form:

THEOREM. *Let $\hat{X}$ be a compactum with* $\dim \hat{X} \leqslant n+1$, *$X$ a closed subset of $\hat{X}$ and let Y be a compactum with* $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(S^n)$. *In order a fundamental sequence $\underline{f}: X \to Y$ be extendable to a fundamental sequence $\underline{\hat{f}}: \hat{X} \to Y$ it is necessary and sufficient that the kernel of the homomorphism $\underline{f}_*: H_n(X) \to H_n(Y)$ contain the kernel of the homomorphism $j_*: H_n(X) \to H_n(\hat{X})$ induced by the inclusion $j: X \to \hat{X}$.*

Proof. Since the necessity of this condition is evident, it remains to prove that it is sufficient. The hypothesis $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(S^n)$ means that there exist two fundamental sequences

$$\underline{\alpha}: Y \to S^n \quad \text{and} \quad \underline{\beta}: S^n \to Y$$

such that $\underline{\beta}\underline{\alpha} \simeq \underline{i}_Y$, where $\underline{i}_Y: Y \to Y$ denotes the fundamental identity sequence. Then $\underline{\alpha}\underline{f}: X \to S^n$ is a fundamental sequence which (since $S^n \in \mathrm{ANR}$) is homotopic to a fundamental sequence generated by a map

$\omega: X \to S^n$. If γ is a true cycle in X being a representative of an element $[\gamma]$ of the kernel of the homomorphism j_*, i.e., if $\gamma \sim 0$ in $\hat{X}$, then $[\gamma]$ belongs to the kernel of the homomorphism $\underline{f}_*$, i.e., $\underline{f}(\gamma) \sim 0$ in Y. It follows that $\underline{\alpha}\underline{f}(\gamma) \sim 0$ in S^n, and hence also $\omega(\gamma) \sim 0$ in S^n. Thus the kernel of the homomorphism $\omega_*: H_n(X) \to H_n(S^n)$ induced by the map ω contains the kernel of the homomorphism j_*. We infer, by the classical Hopf's Extension Theorem, that there exists a map $\hat{\omega}: \hat{X} \to S^n$ which is an extension of the map ω. Then the fundamental sequence $\underline{\hat{\omega}}: \hat{X} \to S^n$ generated by the map $\hat{\omega}$ is an extension of the fundamental sequence $\underline{\omega}$: $X \to S^n$ generated by the map ω. Since $\underline{\omega} \simeq \underline{\alpha}\underline{f}$, we infer, by the Homotopy Extension Theorem for fundamental sequences (proved by Patkowska [3], p. 87), that there exists a fundamental sequence $\underline{\vartheta}: \hat{X} \to S^n$ which is an extension of the fundamental sequence $\underline{\alpha}\underline{f}$ and which satisfies the condition $\underline{\vartheta} \simeq \underline{\hat{\omega}}$. Then the fundamental sequence $\underline{\beta}\underline{\vartheta}: \hat{X} \to Y$ is an extension of the fundamental sequence $\underline{\beta}\underline{\alpha}\underline{f}$. But $\underline{\beta}\underline{\alpha} \simeq \underline{i}_Y$ implies that $\underline{\beta}\underline{\alpha}\underline{f} \simeq \underline{f}$. Using the theorem of Patkowska once again, we infer that there exists a fundamental sequence $\underline{\hat{f}}: \hat{X} \to Y$ which is an extension of the fundamental sequence $\underline{f}$. Thus the proof of the theorem is complete.

PROBLEM. Does the last theorem remain true if we replace the hypothesis $\dim \hat{X} \leqslant n+1$ by the hypothesis that the fundamental dimension of X is not greater than n and the fundamental dimension of $\hat{X}$ is not greater than $n+1$? (**P 927**)

REFERENCES

[1] K. Borsuk, *Concerning the notion of the shape of compacta*, Proceedings of the International Symposium on Topology and its Applications, Herceg-Novi 1968, Beograd 1969, p. 95-104.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1941.

[3] H. Patkowska, *A homotopy extension theorem for fundamental sequences*, Fundamenta Mathematicae 64 (1969), p. 87-89.

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