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SOME LOCAL PROPERTIES OF SUSLINIAN COMPACTA

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By a *compactum* we understand any compact metric space and a *continuum* means a connected compactum. We say that a compactum X is *Suslinian* provided each collection of mutually disjoint non-degenerate continua contained in X is countable. The concept of Suslinian continua has been investigated in [1] and [5]. Observe that, as implied directly by the definition of Suslinian compacta, all but a countable number of open balls with different radii and with a fixed center in a Suslinian compactum X have zero-dimensional or empty boundaries in X. Consequently, each Suslinian compactum is at most one-dimensional.

We use the following notation. Suppose C is a collection of subsets of a space. We denote by $|C|$ the union of all elements of C. Given a metric space X, we define subsets $K(X)$ and $L(X)$ of X by the formulae

$$K(X) = \bigcap_{n=1}^{\infty} |\{\operatorname{Int} F\colon F \subset X, \operatorname{diam} F < n^{-1}, F \text{ connected}\}|, \tag{1}$$

$$L(X) = \bigcap_{n=1}^{\infty} |\{G\colon G \subset X, \operatorname{diam} G < n^{-1}, G \text{ connected open}\}|. \tag{2}$$

In other words, $K(X)$ and $L(X)$ are the sets of all points of X at which X is *connected im kleinen* and *locally connected*, respectively.

1. Preliminaries. It follows from (1) and (2) that both $K(X)$ and $L(X)$ are G_δ-sets in X.

1.1. *If X is a metric space, then* $\operatorname{Int} K(X) \subset L(X) \subset K(X)$.

Proof. The inclusion $L(X) \subset K(X)$ trivially holds. On the other hand, it is easy to see that all components of an open subset of X which is contained in $K(X)$ are open (compare [4], p. 230). Thus, given a point p belonging to the interior of $K(X)$, the components of sufficiently small open balls having p as the center are connected open sets, which means that $p \in L(X)$, by (2).

Proof. Let G be a non-empty open subset of X. We define inductively an infinite sequence of continua $K_n \subset X$ such that, for $n = 1, 2, \ldots,$

$$(4) \qquad \emptyset \neq K_{n+1} \subset \operatorname{Int} K_n \subset G,$$

$$(5) \qquad \operatorname{diam} K_n < n^{-1}.$$

First, we can assume X to be non-degenerate; otherwise, the theorem is trivial. Then there exists an open ball B_1 with a radius $\varepsilon_1 < \frac{1}{2}$ such that the closure C_1 of B_1 is a proper subset of X and $C_1 \subset G$. Let $\boldsymbol{K}_1$ denote the collection of all components of C_1 which intersect B_1. Each element of $\boldsymbol{K}_1$ also intersects the boundary of C_1 in X (see [4], p. 172), so that $\boldsymbol{K}_1$ is a collection of non-degenerate continua. Since X is Suslinian, $\boldsymbol{K}_1$ is countable and, clearly, B_1 is contained in $|\boldsymbol{K}_1|$. By the Baire theorem, there exists at least one element $K_1 \in \boldsymbol{K}_1$ such that the interior of $B_1 \cap K_1$ in B_1 is non-empty. Thus $\operatorname{Int} K_1 \neq \emptyset$. Moreover, we have $K_1 \subset G$ and

$$\operatorname{diam} K_1 \leqslant \operatorname{diam} C_1 = \operatorname{diam} B_1 \leqslant 2\varepsilon_1 < 1.$$

Assume now K_n is defined so that $\operatorname{Int} K_n \neq \emptyset$. We find a continuum K_{n+1} in exactly the same manner as we found the continuum K_1 above. One has only to replace G, B_1, ε_1, $\frac{1}{2}$, C_1, $\boldsymbol{K}_1$ and K_1 by $\operatorname{Int} K_n$, B_{n+1}, ε_{n+1}, $\frac{1}{2}(n+1)^{-1}$, C_{n+1}, $\boldsymbol{K}_{n+1}$ and K_{n+1}, respectively. Thus $\operatorname{Int} K_{n+1} \neq \emptyset$, and both conditions (4) and (5) are satisfied.

According to (4), there exists a point p which belongs to all continua K_n $(n = 1, 2, \ldots)$. This point belongs also to $\operatorname{Int} K_n$ $(n = 1, 2, \ldots)$. By (1) and (5), we get $p \in K(X)$, and the proof of 3.1 is complete, since $p \in G$ as well.

Remark. By 1.1, a stronger theorem than 3.1 would hold if one could put $L(X)$ in place of $K(X)$. This stronger version of 3.1, however, is false as can be seen from the example described in 3.2.

Let us recall that a compactum is said to be *rational* provided it has an open basis composed entirely of sets whose boundaries are countable. It is rather apparent that each rational compactum is Suslinian, but the reverse implication is not true (see [5], p. 132 and 135). By a *dendroid* we mean any arcwise connected continuum such that the common part of each two continua contained in it is a continuum. Each dendroid is at most one-dimensional and *acyclic*, i. e. all continuous mappings of it into the circle are homotopic to a constant mapping. Some properties of the following example have been possessed by an earlier example of a continuum constructed by Grace [3].

3.2. Example. *There exists a non-degenerate rational dendroid X in the plane such that each non-empty connected open subset of X is dense in X*; hence $L(X) = \emptyset$.

Proof. In this construction we use rhombi with oblique angles, each rhombus being considered to be a compact 2-cell in the plane. We also denote by $\overline{pq}$ the straight line closed segment with end-points p and q. Let (p, q, r) be an ordered triple of non-collinear points of the plane, and let T denote the triangle with vertices p, q and r. We first define a countable collection $\boldsymbol{R}(p, q, r)$ of rhombi which form a sequence converging to $\overline{pq}$ and which all are contained in T. Namely, take a sequence of distinct points $q_1, q_2, \ldots$ satisfying the conditions

$$\lim_{i\to\infty} q_i = q, \qquad q_i \in \overline{qr}\setminus\{q, r\}, \qquad \operatorname{dist}(q_i, q) \leqslant \tfrac{1}{2}\operatorname{dist}(p, q),$$
$$\text{for } i = 1, 2, \ldots$$

Then choose a sequence of rhombi $R_1, R_2, \ldots$ in T such that $\overline{pq_i}$ is the longer diagonal of R_i, the intersection of R_i with the boundary of T is $\{p, q_i\}$, and $R_i \cap R_j = \{p\}$ for $i, j = 1, 2, \ldots$ and $i \neq j$. We put

$$\boldsymbol{R}(p, q, r) = \{R_i\colon i = 1, 2, \ldots\}.$$

Next, for each rhombus R, we define a countable collection $\boldsymbol{R}(R)$ of rhombi in the following way. Let $\overline{pq}$ and $\overline{rs}$ be the longer diagonal and the shorter diagonal of R, respectively. Let o be the intersection point of $\overline{pq}$ and $\overline{rs}$. The collection $\boldsymbol{R}(R)$ is defined by

$$\boldsymbol{R}(R) = \boldsymbol{R}(o, p, r) \cup \boldsymbol{R}(o, q, s) \cup \boldsymbol{R}(p, o, s) \cup \boldsymbol{R}(q, o, r).$$

Given a rhombus R_0, we now define collections $\boldsymbol{R}_n$ $(n = 0, 1, \ldots)$ of rhombi inductively by setting

$$\boldsymbol{R}_0 = \{R_0\}, \qquad \boldsymbol{R}_{n+1} = \bigcup_{R\in\boldsymbol{R}_n} \boldsymbol{R}(R),$$

and let

$$X = \bigcap_{n=0}^{\infty} \operatorname{cl}|\boldsymbol{R}_n|.$$

The end-points of the longer diagonal of R_0 belong to each of the sets $|\boldsymbol{R}_n|$ and it is not difficult to observe that the closures of these sets form a decreasing sequence of continua; so that X is a non-degenerate continuum in the plane. Moreover, the set

$$D_n = \operatorname{cl}(X\setminus|\boldsymbol{R}_{n+1}|)$$

is the union of all the longer diagonals of rhombi belonging to $\boldsymbol{R}_0 \cup \ldots \cup \boldsymbol{R}^n$ and one sees rather easily that D_n is a dendroid $(n = 0, 1, \ldots)$. The intersection of each rhombus $R \in \boldsymbol{R}_{n+1}$ with D_n is a degenerate set composed of an end-point of the longer diagonal of R. Also, the same end-point is the only point at which R can meet any other rhombus of $\boldsymbol{R}_{n+1}$. We conclude that this vertex of R cuts the continuum $\mathrm{cl}\,|\boldsymbol{R}_{n+1}|$ between some two points of the set D_{n+1} and, consequently, it is also a cut-point of X. On the other hand, each rhombus R_i of $\boldsymbol{R}(p, q, r)$ has the diameter

$$\operatorname{diam} R_i = \operatorname{dist}(p, q_i) \leqslant \operatorname{dist}(p, q) + \operatorname{dist}(q_i, q) \leqslant \frac{3}{2}\operatorname{dist}(p, q),$$

which implies that $\operatorname{diam} R' \leqslant \frac{3}{4}\operatorname{diam} R$ for $R' \in \boldsymbol{R}(R)$. As a result, the diameters of rhombi belonging to $\boldsymbol{R}_n$ converge to zero when n tends to infinity. It follows that the continuum X has a dense subset of cut-points and, therefore, no disk is contained in X; this means X is one-dimensional.

The continua $\mathrm{cl}\,|\boldsymbol{R}_n|$ $(n = 0, 1, \ldots)$ do not cut the plane, which one can readily check by using their definition. Hence X does not cut the plane either. Since X is one-dimensional, the common part of each two continua contained in X must then be a continuum (compare [4], p. 505-506). To see that X is arcwise connected, let us select two points $x, y \in X$ and let A_n be an arc joining x and y in $\mathrm{cl}\,|\boldsymbol{R}_n|$. The set $A_{n+1} \cap D_n$, if non-degenerate, is a subarc of A_{n+1}, for $n = 0, 1, \ldots$ Since the common part of each rhombus of $\boldsymbol{R}_{n+1}$ with any other rhombus of $\boldsymbol{R}_{n+1}$ or with D_n is degenerate, we notice that A_{n+1} can differ from A_n only on the part contained in the rhombi of $\boldsymbol{R}_n$ to which x or y may belong. The diameters of those rhombi converging to zero when n tends to infinity, the arcs A_n converge to an arc which joins x and y in X. Thus X is a dendroid.

The second important property of X, i. e. that of being rational, will be demonstrated if we prove the existence of a countable subset $Q \subset X$ such that $X \setminus Q$ is totally disconnected (see [6], p. 95). Let $\beta_0, \beta_1, \ldots$ be a sequence composed of all the numbers $t \in [0, 1]$ of type $t = 2^{-k}l$, where k and l are integers. We assume $\beta_0 = 0$ and $\beta_1 = 1$. For each rhombus $R \in \boldsymbol{R}_n$ $(n = 0, 1, \ldots)$, we denote by $q_0(R)$ and $q_1(R)$ the end-points of the longer diagonal of R, and we take points $q_m(R)$ of this diagonal such that

$$\operatorname{dist}[q_m(R), q_0(R)] = \beta_m \operatorname{dist}[q_0(R), q_1(R)] \quad \text{for } m = 0, 1, \ldots$$

Then the set

$$Q = \bigcup_{m=0}^{\infty} \bigcup_{n=0}^{\infty} \{q_m(R)\colon R \in \boldsymbol{R}_n\}$$

is countable and $Q \subset X$. Let $x, y \in X \setminus Q$ be two points arbitrarily chosen. In order to show that $X \setminus Q$ is totally disconnected, we have to prove that x and y belong to two different quasi-components of $X \setminus Q$ or, which is the same, that $X \setminus Q$ is not connected between x and y. The following two cases are to be distinguished.

Case 1. For each $n = 0, 1, \ldots$, at least one of the points x, y belongs to $|\boldsymbol{R}_n|$. Let n_0 be large enough so that the diameter of each rhombus of $\boldsymbol{R}_{n_0}$ is less than $\operatorname{dist}(x, y)$, and let $R \in \boldsymbol{R}_{n_0}$ be a rhombus containing one of the points x, y, say $x \in R$. Thus $y \notin R$ and either $q_0(R)$ or $q_1(R)$ cuts the continuum $\mathrm{cl}|\boldsymbol{R}_{n_0}|$ between x and y. This cut-point belongs to Q and it also cuts X between the latter two points, so that $X \setminus Q$ certainly is not connected between them.

Case 2. There exists an $n_1 = 0, 1, \ldots$ such that none of the points x, y belongs to $|\boldsymbol{R}_{n_1}|$. Then $n_1 > 0$ and we can assume that n_1 is the minimum integer having these properties. Consequently, there exists a rhombus $R' \in \boldsymbol{R}_{n_1-1}$ which contains at least one of the points x, y, say $x \in R'$. If $y \notin R'$, the argument as in Case 1 can be adopted to show that $X \setminus Q$ is not connected between x and y. We therefore assume $y \in R'$, which implies that both x and y belong to the longer diagonal of R', since neither x nor y belongs to any rhombus of $\boldsymbol{R}(R') \subset \boldsymbol{R}_{n_1}$. And we are going to indicate which subset of Q cuts X between x and y under these circumstances. Let

$$\boldsymbol{R}'_0 = \{R'\}, \qquad \boldsymbol{R}'_{k+1} = \bigcup_{R \in \boldsymbol{R}'_k} \boldsymbol{R}(R) \quad \text{for } k = 0, 1, \ldots$$

and let $C_k = \mathrm{cl}\, E_k$, where E_k denotes the set of all the end-points of longer diagonals of rhombi belonging to $\boldsymbol{R}'_k$. Then $\boldsymbol{R}'_k \subset \boldsymbol{R}_{k+n_1-1}$ and $E_k \subset Q$ $(k = 0, 1, \ldots)$. Moreover, it follows directly from the definition of $\boldsymbol{R}(R)$ that the only cluster points of E_k which do not belong to E_k are the mid--points of halves of longer diagonals of rhombi of $\boldsymbol{R}'_{k-2}$, the mid-points of halves and quarters of longer diagonals of rhombi of $\boldsymbol{R}'_{k-3}$, etc. Those mid-points, however, all belong to Q, whence $C_k \subset Q$ for $k = 0, 1, \ldots$ Now, observe that the common part of C_k and the longer diagonal of R' consists of the end-points of 2^k congruent intervals which form a dyadic subdivision of the diagonal. The set C_k cuts X between each two points taken from different intervals of this subdivision. Since the length of these intervals converges to zero when k tends to infinity, there exists an integer $k_0 > 0$ such that the points x and y are located in different two out of 2^{k_0} intervals, and thus C_{k_0} cuts X between them.

Finally, to prove the last property of X as stated in 3.2, let us consider a non-empty connected open subset G of X. The set

$$E = \bigcup_{m=0}^{1} \bigcup_{n=0}^{\infty} \{q_m(R) \colon R \in \boldsymbol{R}_n\}$$

consists of all the end-points of longer diagonals of rhombi belonging to $\boldsymbol{R}_0 \cup \boldsymbol{R}_1 \cup \ldots$ Since the diameters of rhombi selected from $\boldsymbol{R}_n$ converge to zero when n increases, we see that E is a dense subset of X and the proof will be completed if we show that $E \subset G$.

Let $R \in \boldsymbol{R}_n$ be a rhombus $(n = 0, 1, \ldots)$ whose longer diagonal contains a point of G. The points

$$p = q_0(R), \qquad q = q_1(R)$$

are the end-points of that diagonal; let o be the mid-point of $\overline{pq}$ and let $\overline{rs}$ be the shorter diagonal of R so that this notation agrees with that already used by us before in the definition of $\boldsymbol{R}(R)$. Then G meets $\overline{op}$ or $\overline{oq}$, say $\overline{op}$. We claim that $o \in G$. Suppose, on the contrary, that $o \notin G$. There exists a point $z \in G \cap \overline{op}$ and, according to the definition of $\boldsymbol{R}(o, p, r)$, the longer diagonals of rhombi of $\boldsymbol{R}(o, p, r)$ contain points which converge to z. Those points belong to X and, consequently, all but a finite number of them must be in G, the set G being open in X. However, the point o is the end-point of the longer diagonal of each rhombus taken from the collection

$$\boldsymbol{R}(o, p, r) \dot{\subset} \boldsymbol{R}(R) \subset \boldsymbol{R}_{n+1}$$

and o cuts the continuum $\mathrm{cl}|\boldsymbol{R}_{n+1}|$ between z and each of those points converging to z. The set G is contained in the latter continuum, which contradicts the assumption that G is connected. Hence $o \in G$. If G meets $\overline{oq}$, the same argument, with p and $\boldsymbol{R}(o, p, r)$ replaced by q and $\boldsymbol{R}(o, q, s)$, respectively, shows that $o \in G$ again.

In a similar manner, using the remaining two collections from the definition of $\boldsymbol{R}(R)$, that is $\boldsymbol{R}(p, o, s)$ and $\boldsymbol{R}(q, o, r)$, one can show that $p \in G$ and $q \in G$. We conclude that if the longer diagonal of a rhombus $R \in \boldsymbol{R}_n$ $(n = 0, 1, \ldots)$ has a point belonging to G, then both the end-points and the mid-point of that diagonal of R belong to G.

Since $E \subset X$ is dense and $G \subset X$ is open and non-empty, there is a point $e_0 \in E \cap G$. Let $e \in E$ be an arbitrary point different from e_0. Clearly, we have $E \subset D_0 \cup D_1 \cup \ldots$ and the dendroids D_n form an increasing sequence $D_0 \subset D_1 \subset \ldots$ There exists an integer j_0 such that D_{j_0} contains e_0 and e. Let $A \subset D_{j_0}$ be the arc joining e_0 and e. The dendroid D_{j_0} being composed entirely of straight line segments, the arc A is a polygonal arc and, moreover, A is the union of a finite chain of the longer diagonals or of the halves of longer diagonals of some rhombi which belong to $\boldsymbol{R}_0 \cup \ldots \ldots \cup \boldsymbol{R}_{j_0}$. Since e_0 is the first end-point of the segments in A and $e_0 \in G$, we conclude by a finite induction that the junction points of these segments as well as the last end-point e all belong to G. Thus $E \subset G$.

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