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SOME INTEGRAL INEQUALITIES OF HARDY TYPE

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In the present note we apply the method used in [6] to obtain certain integral inequalities of Hardy type, i.e. integral inequalities of the form

$$\int_I s|h|^p dt \leqslant \int_I r|\dot{h}|^p dt, \tag{1}$$

where $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, $\dot{h} \equiv dh/dt$ and $p > 1$. Inequalities of the form (1) have been considered by Beesack [2], [3], Redheffer [10], Benson [4], Boyd [5], Tomaseli [12] and others (for an extensive bibliography see [3] and [9]).

Let us denote by *abs C* the class of real functions which are defined and absolutely continuous on the open interval $I = (\alpha, \beta)$, $-\infty \leqslant \alpha < \beta \leqslant \infty$. Let p be any real number such that $p > 1$ and let $r \in abs\,C$ and $\varphi \in abs\,C$ be functions such that $r > 0$, $\varphi > 0$ in I and $r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi} \in abs\,C$. Let us put

$$s \equiv -(r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi})^{\cdot}\varphi^{1-p} \quad \text{and} \quad v \equiv r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi}\cdot\varphi^{1-p}.$$

LEMMA 1. *For every function $h \in abs\,C$ the equality*

$$r|\dot{h}|^p = s|h|^p + g + (v|h|^p)^{\cdot} \tag{2}$$

holds almost everywhere in the interval I with $g \geqslant 0$. Moreover, $g = 0$ if and only if $(\varphi^{-1}h)^{\cdot} = 0$.

Proof. It follows from the assumptions that $v|h|^p \in abs\,C$. Let $g = r|\dot{h}|^p - s|h|^p - (v|h|^p)^{\cdot}$ a.e. in the interval I. If we substitute $h = \varphi f$, where $f \in abs\,C$, then this equality can easily be transformed as

$$g = r\{|\varphi\dot{f} + \dot{\varphi}f|^p - |\dot{\varphi}f|^p - p\varphi\dot{f}|\dot{\varphi}f|^{p-1}\operatorname{sgn}(\dot{\varphi}f)\} \equiv r\omega,$$

ω being either of the form $\omega = (A+B)^p - B^p - pAB^{p-1}$, where $A+B \geqslant 0$ and $B \geqslant 0$, or of the form $\omega = (A-B)^p - B^p + pAB^{p-1}$, where $A-B \geqslant 0$ and $B \geqslant 0$. Here we put $A = \varphi\dot{f}$ or $A = -\varphi\dot{f}$ and $B = |\dot{\varphi}f|$.

In both cases we have $\omega \geqslant 0$ and $\omega = 0$ if and only if $A = 0$ (see [8], Theorem 41, and [1]). Then $g \geqslant 0$ since $r > 0$, and $g = 0$ if and only if $\varphi f = 0$, i.e. $(\varphi^{-1}h)^{\cdot} = 0$ for $\varphi > 0$.

Let us denote by $\hat{H}$ the class of functions $h \in abs\, C$ satisfying the following integral and limit conditions:

$$\int_I s|h|^p dt > -\infty, \qquad \int_I r|\dot{h}|^p dt < \infty; \tag{3}$$

$$\liminf_{t\to\alpha} v|h|^p < \infty, \qquad \limsup_{t\to\beta} v|h|^p > -\infty. \tag{4}$$

THEOREM 1. *For every function $h \in \hat{H}$ the following statements are valid*:

(i) *Both limits in* (4) *are proper and finite.*

(ii) *The inequality*

$$\lim_{t\to\beta} v|h|^p - \lim_{t\to\alpha} v|h|^p \leqslant \int_I (r|\dot{h}|^p - s|h|^p)\,dt \tag{5}$$

holds.

(iii) *If $\varphi \notin \hat{H}$ and $h \not\equiv 0$, then* (5) *is a strict inequality.*

(iv) *If $\varphi \in \hat{H}$, then* (5) *becomes an equality only in case where $h = c\varphi$ with c = const.*

Proof. Let $h \in \hat{H}$. Since $r|\dot{h}|^p \geqslant 0$, it follows from (3) that the function $r|\dot{h}|^p$ is summable in I. Now we prove that the functions $s|h|^p$ and $(v|h|^p)^{\cdot}$ are summable in every interval $\langle a, b\rangle$, where $\alpha < a < b < \beta$. We have

$$s|h|^p = (r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi})^{\cdot}\,\varphi^{1-p}|h|^p$$

and we notice that $\varphi^{1-p}|h|^p$ is measurable and bounded in $\langle a, b\rangle$, and the function $(r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi})^{\cdot}$ is summable in $\langle a, b\rangle$, since $r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi} \in abs\, C$ by assumption. Thus $s|h|^p$ is summable in $\langle a, b\rangle$. Also $(v|h|^p)^{\cdot}$ is summable in $\langle a, b\rangle$, since $v|h|^p \in abs\, C$.

Let $g = r|\dot{h}|^p - s|h|^p - (v|h|^p)^{\cdot}$ a.e. in I. The function g is summable in $\langle a, b\rangle$ and the equality

$$\int_a^b r|\dot{h}|^p dt = \int_a^b s|h|^p dt + \int_a^b g\,dt + v|h|^p\Big|_a^b \tag{6}$$

holds.

Now, by (4), there exist two sequences $\{a_n\}$ and $\{b_n\}$ such that $\alpha < a_n < b_n < \beta$, $a_n \to \alpha$, $b_n \to \beta$ and

$$\lim_{n\to\infty} v|h|^p\big|_{a_n} < \infty, \qquad \lim_{n\to\infty}(-v|h|^p)\big|_{b_n} < \infty.$$

Thus there exists some finite constant C such that

$$-v|h|^p\Big|_{a_n}^{b_n} < C < \infty.$$

From this condition and from (6) it follows that

$$\int_{a_n}^{b_n} s|h|^p dt < \int_{a_n}^{b_n} r|\dot{h}|^p dt + C,$$

since $g \geqslant 0$ by Lemma 1. Further we have

$$\lim_{n\to\infty} \int_{a_n}^{b_n} s|h|^p dt = \int_I s|h|^p dt,$$

since by assumption the integral $\int_I s|h|^p dt$ exists and the function $s|h|^p$ is summable in any interval $\langle a, b\rangle \subset I$. Hence we get the bound

$$\int_I s|h|^p dt < \int_I r|\dot{h}|^p dt + C < \infty.$$

From the first condition of (3) we conclude that the function $s|h|^p$ is summable in the whole interval I.

In a similar way, using (6), we prove that g is summable in I.

We show that all integrals in (6) have finite limits as $a \to \alpha$ or $b \to \beta$. Hence (i) of the theorem follows immediately.

By (6), as $a \to \alpha$ and $b \to \beta$, we get the equality

$$\int_I r|\dot{h}|^p dt = \int_I s|h|^p dt + \int_I g\, dt + \lim_{t\to\beta} v|h|^p - \lim_{t\to\alpha} v|h|^p, \tag{7}$$

whence we obtain (5), since $g \geqslant 0$ in I by Lemma 1.

From (7) it follows that (5) becomes an equality for some non-vanishing function $h \in \hat{H}$ if and only if

$$\int_I g\, dt = 0,$$

i.e. $g = 0$ (a.e.) in I, as $g \geqslant 0$. Due to Lemma 1 we have $g = 0$ if and only if $(\varphi^{-1}h)^{\cdot} = 0$. On the other hand, $\varphi^{-1}h \in abs\, C$, and hence $h = c\varphi$, where $c = \mathrm{const} \neq 0$. Consequently, $\varphi \in \hat{H}$. Now it is easy to complete the proof of (iii) and (iv).

Now we give necessary conditions for φ in order that inequality (5) may become an equality for some non-vanishing function h.

LEMMA 2. *The function φ belongs to the class $\hat{H}$ if and only if the following two conditions are satisfied*:

(i)
$$\int_I r|\dot{\varphi}|^p dt < \infty;$$

(ii)
$$\int_I |s|\varphi^p dt < \infty.$$

Condition (ii) *may be replaced by the conjunction of the following two conditions*:

(iii) $$\int_I s^- \varphi^p dt < \infty, \quad \textit{where } s^- = \max(-s, 0);$$

(iv) *there exist finite limits of the expression* $r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi\cdot\varphi$ *as* $t \to a$ *and* $t \to \beta$.

Proof. Let $\varphi \in \hat H$. Condition (i) coincides with the second of conditions (3). We obtain (ii) directly from Theorem 1, where we have proved that $s|h|^p$ is summable in I provided $h \in \hat H$.

Now, assume that φ satisfies (i) and (ii). Condition (iii) follows immediately from (ii). By Lemma 1 for $h = \varphi$ we have $g = 0$ and from (6) we get the identity

$$r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi\cdot\varphi\Big|_a^b = \int_a^b r|\dot\varphi|^p dt - \int_a^b s\varphi^p dt,$$

where $a < a < b < \beta$. Hence (iv) is fulfilled.

Finally, assume that φ satisfies conditions (i), (iii) and (iv). Then φ satisfies (3) and (4), i.e. $\varphi \in \hat H$.

Let us note that in the often appearing case of $s \geqslant 0$ condition (iii) is trivially satisfied.

Let H be a class of functions $h \in abs\, C$ satisfying the integral conditions (3), the limit conditions (4) and the limit condition

(8) $$\liminf_{t\to a} v|h|^p \leqslant \limsup_{t\to\beta} v|h|^p.$$

Obviously, $H \subset \hat H$. By Theorem 1, conditions (4) and (8) may be written in the equivalent forms

(4′) $$\lim_{t\to a} v|h|^p < \infty, \quad \lim_{t\to\beta} v|h|^p > -\infty,$$

(8′) $$\lim_{t\to a} v|h|^p \leqslant \lim_{t\to\beta} v|h|^p.$$

THEOREM 2. *For every function* $h \in H$ *the inequality*

(9) $$\int_I s|h|^p dt \leqslant \int_I r|\dot h|^p dt$$

holds. If $h \not\equiv 0$, *then* (9) *becomes an equality if and only if* $\varphi^{-1}h = \text{const} \neq 0$, *and the additional conditions*

(10) $$\varphi \in \hat H, \quad \lim_{t\to a} r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi\cdot\varphi = \lim_{t\to\beta} r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi\cdot\varphi$$

are satisfied.

Proof. By virtue of (8′) and Theorem 1, inequality (9) follows from (5). If both sides of (9) are equal for some non-vanishing function $h \in H$, then by (5) and (8′) we have

$$\lim_{t\to a} v|h|^p = \lim_{t\to\beta} v|h|^p.$$

Applying Theorem 1 once again we get $\varphi \in \hat{H}$ and $h = c\varphi$, where $c = \text{const} \neq 0$. This proves the validity of (10), since $v\varphi^p = r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi\cdot\varphi$. The theorem follows now easily.

Inequalities of form (9), which do not contain explicitly the limit conditions actually, are said to be of *Hardy type* (cf. [3]). Let us note that the condition $\varphi \in H$ is not sufficient for inequality (9) to become an equality.

Now we deal with the class H in the especially interesting case of $s > 0$ a.e. in the interval I. In the sequel, let $q = p/(p-1)$.

LEMMA 3. *Let $h \in \operatorname{abs} C$ and let $\int_I r|\dot h|^p dt < \infty$. If*

$$\int_a^t r^{-q/p} dt < \infty \qquad \left(\textit{resp.}\ \int_t^\beta r^{-q/p} dt < \infty\right)$$

for some $t \in I$, then there exists a finite limit value

$$h(a) = \lim_{t\to a} h \qquad \left(\textit{resp.}\ h(\beta) = \lim_{t\to\beta} h\right).$$

Proof. Using Hölder's inequality we obtain the estimation

$$(11)\qquad |h(b)-h(a)| = \left|\int_a^b \dot h\, dt\right| \leqslant \int_a^b |\dot h|\, dt \leqslant \left(\int_a^b r^{-q/p} dt\right)^{1/q}\left(\int_a^b r|\dot h|^p dt\right)^{1/p},$$

where $a < a < b < \beta$. Lemma 3 follows now from the Cauchy condition for the existence of the limit.

LEMMA 4. *Let $s > 0$ almost everywhere in the interval I.*

(i) *There exist limit values $v(a) = \lim_{t\to a} v$ and $v(\beta) = \lim_{t\to\beta} v$ and $v(a) > v(\beta)$.*

(ii) *If $v(a) \neq 0$ (resp. $v(\beta) \neq 0$), then*

$$\int_a^t r^{-q/p} dt < \infty \qquad \left(\textit{resp.}\ \int_t^\beta r^{-q/p} dt < \infty\right)$$

for some $t \in I$ and

$$v\left(\int_a^t r^{-q/p} dt\right)^{p/q} = O(1) \qquad \textit{as } t \to a$$

$$\left(\textit{resp.}\ v\left(\int_t^\beta r^{-q/p} dt\right)^{p/q} = O(1) \ \textit{as}\ t \to \beta\right).$$

Proof. The function v belongs to the class *abs C* and satisfies the differential equation

$$\dot{v}+\frac{p}{q}r^{-q/p}|v|^{q}+s=0 \tag{12}$$

a.e. in I. Thus we have $-\dot{v}\geqslant s$, and integrating this inequality we find

$$v(a)-v(b)\geqslant\int_a^b s\,dt>0\quad \text{for } \alpha<a<b<\beta,$$

i.e. v is decreasing in I, which proves (i).

Let $v(\alpha)\neq 0$ and let us consider a neighbourhood of α in which $v\neq 0$. We denote this neighbourhood by U. By (12) we have the estimation

$$\int_a^t r^{-q/p}dt\leqslant\frac{q}{p}\int_a^t |v|^{-q}\dot{v}\,dt=|v|^{-q/p}\operatorname{sgn}v-|v(a)|^{-q/p}\operatorname{sgn}v(a) \tag{13}$$

for $\alpha<a<t<\beta$ in U. If $v(\alpha)>0$ (i.e. $v>0$ in U), then by (13) as $a\to\alpha$ we have

$$\int_\alpha^t r^{-q/p}dt\leqslant v^{-q/p}\quad\text{and}\quad 0\leqslant v\Big(\int_\alpha^t r^{-q/p}dt\Big)^{p/q}\leqslant 1.$$

If $v(\alpha)<0$ (i.e. $v<0$ in U), then by (13) as $a\to\alpha$ we obtain

$$\int_\alpha^t r^{-q/p}dt\leqslant|v(\alpha)|^{-q/p}\quad\text{and}\quad v|v(\alpha)|^{-1}\leqslant v\Big(\int_\alpha^t r^{-q/p}dt\Big)^{p/q}\leqslant 0.$$

Thus in both cases we get

$$\int_\alpha^t r^{-q/p}dt<\infty\quad\text{and}\quad v\Big(\int_\alpha^t r^{-q/p}dt\Big)^{p/q}=O(1)\quad\text{as } t\to\alpha,$$

which proves (ii).

We denote by H_0 (resp. H^0) the class of functions $h\in$ *abs C* satisfying the integral condition

$$\int_I r|\dot{h}|^p dt<\infty \tag{14}$$

and the limit condition

$$\liminf_{t\to\alpha}|h|=0\quad(\text{resp. }\liminf_{t\to\beta}|h|=0). \tag{15}$$

In the sequel, (15) is equivalent to

$$h(\alpha)\equiv\lim_{t\to\alpha}h=0\quad(\text{resp. }h(\beta)\equiv\lim_{t\to\beta}h=0). \tag{15'}$$

THEOREM 3. *Let $s > 0$ almost everywhere in the interval I.*

(i) *If $v(a) = \infty$ and $v(\beta) \geqslant 0$, then $H = H_0$.*

(ii) *If $0 < v(a) < \infty$ and $v(\beta) \geqslant 0$, then $H \supset H_0$.*

(iii) *If $v(a) \leqslant 0$ and $v(\beta) = -\infty$, then $H = H^0$.*

(iv) *If $v(a) \leqslant 0$ and $-\infty < v(\beta) < 0$, then $H \supset H^0$.*

(v) *If $0 < v(a) \leqslant \infty$ and $-\infty \leqslant v(\beta) < 0$, then $H = H_0 \cap H^0$.*

Proof. Let $v(a) = \infty$ and let $h \in H$. Then by Lemmas 4 and 3 there exists a finite limit value

$$h(a) = \lim_{t \to a} h.$$

If $h(a) \neq 0$, then

$$\lim_{t \to a} v|h|^p = \infty,$$

which contradicts (4′). Thus $h(a) = 0$, i.e. $h \in H_0$. Similarly, we prove that if $v(\beta) = -\infty$ and $h \in H$, then $h \in H^0$.

Let $0 < v(a) < \infty$, $-\infty < v(\beta) < 0$, and $h \in H$. Then by Lemmas 4 and 3 the values $h(a)$ and $h(\beta)$ are finite. Further, we have

$$\lim_{t \to a} v|h|^p = v(a)|h(a)|^p \geqslant 0 \quad \text{and} \quad \lim_{t \to \beta} v|h|^p = v(\beta)|h(\beta)|^p \leqslant 0,$$

and by (8′) we obtain $v(a)|h(a)|^p = v(\beta)|h(\beta)|^p = 0$. Hence, it follows that $h(a) = h(\beta) = 0$, i.e. $h \in H_0 \cap H^0$.

Let $v(a) > 0$ and $h \in H_0$. Then $h(a) = 0$ and we have the estimation

$$0 \leqslant v|h|^p \leqslant v\Big(\int_a^t r^{-q/p} dt\Big)^{p/q} \int_a^t r|\dot{h}|^p dt, \tag{16}$$

which follows from (11) for $a \to a$ and $b = t$. Hence, by Lemma 4 and condition (14), it follows that

$$\lim_{t \to a} v|h|^p = 0.$$

Similarly, if $v(\beta) < 0$ and $h \in H^0$, then

$$\lim_{t \to \beta} v|h|^p = 0.$$

Let $v(a) > 0$, $v(\beta) \geqslant 0$, and $h \in H_0$. Then

$$\lim_{t \to a} v|h|^p = 0 \quad \text{and} \quad \lim_{t \to \beta} v|h|^p \geqslant 0,$$

so that (8′) is satisfied. Hence we infer that $h \in H$. Similarly, if $v(a) \leqslant 0$, $v(\beta) < 0$ and $h \in H^0$, then $h \in H$.

Finally, let $0 < v(a) < \infty$, $-\infty < v(\beta) < 0$, and $h \in H_0 \cap H^0$. Then we have

$$\lim_{t \to a} v|h|^p = 0 \quad \text{and} \quad \lim_{t \to \beta} v|h|^p = 0,$$

whence $h \in H$.

Remark. In [3] inequality (9) is studied for $h \in H_0$ (resp. $h \in H^0$) under the additional assumption that

$$v\Big(\int_a^t r^{-q/p}dt\Big)^{p/q} = O(1) \quad \text{as } t \to a$$

$$\text{(resp. } v\Big(\int_t^\beta r^{-q/p}dt\Big)^{p/q} = O(1) \text{ as } t \to \beta).$$

By Lemmas 3 and 4 this assumption may be omitted (it follows from other assumptions).

Theorem 3 in cases (i)-(iv) may be strengthened as follows:

THEOREM 4. *Let $s > 0$ almost everywhere in the interval I.*

(i) *If $v(\beta) \geqslant 0$, then for every function $h \in H_0$ the inequality*

$$\int_I s|h|^p dt + \lim_{t\to\beta} v|h|^p \leqslant \int_I r|\dot{h}|^p dt \tag{17}$$

holds. If $h \not\equiv 0$, then (17) *becomes an equality if and only if $\varphi^{-1}h = \mathrm{const} \neq 0$ and the additional conditions*

$$\varphi \in H_0, \quad \lim_{t\to\beta} r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi}\cdot\varphi = 0 \tag{18}$$

are satisfied. If $v(\beta) > 0$, then there exists a finite limit value

$$h(\beta) = \lim_{t\to\beta} h$$

and (17) *takes the form*

$$\int_I s|h|^p dt + v(\beta)|h(\beta)|^p \leqslant \int_I r|\dot{h}|^p dt. \tag{17'}$$

(ii) *If $v(a) \leqslant 0$, then for every function $h \in H^0$ the inequality*

$$\int_I s|h|^p dt - \lim_{t\to a} v|h|^p \leqslant \int_I r|\dot{h}|^p dt \tag{19}$$

holds. If $h \not\equiv 0$, then (19) *becomes an equality if and only if $\varphi^{-1}h = \mathrm{const} \neq 0$ and the additional conditions*

$$\varphi \in H^0, \quad \lim_{t\to a} r|\dot{\varphi}|^{p-1}\operatorname{sgn}\dot{\varphi}\cdot\varphi = 0 \tag{20}$$

are satisfied. If $v(a) < 0$, then there exists a finite limit value

$$h(a) = \lim_{t\to a} h$$

and (19) *takes the form*

$$\int_I s|h|^p dt - v(a)|h(a)|^p \leqslant \int_I r|\dot{h}|^p dt. \tag{19'}$$

Proof. We prove only (i) (case (ii) is to be proved similarly).

Let $v(\beta) \geqslant 0$ and let $h \in H_0$. Then by Lemma 4 (i) we have $v(a) > 0$. Thus (16) holds. Further, by (16) and Lemma 4 (ii) we get

$$\lim_{t \to a} v|h|^p = 0.$$

Obviously, $h \in \hat{H}$, since $H_0 \subset H \subset \hat{H}$ (Theorem 3 (i) and (ii)). Hence, by Theorem 1 (ii), from (5) we obtain (17).

If $h \not\equiv 0$, then by Theorem 1 (iii) and (iv) and by Lemma 2, we get immediately the conditions for the equality to hold in (17).

If $v(\beta) > 0$, then by Lemma 4 (ii) we have

$$\int_t^\beta r^{-q/p} dt < \infty,$$

and by Lemma 3 there exists a finite limit value

$$h(\beta) = \lim_{t \to \beta} h,$$

which completes the proof.

Example 1. Let us take $I = (a, \beta)$ with $0 \leqslant a < \beta \leqslant \infty$, $r = t^{p-a}$ and $\varphi = t^{(a-1)/p}$, where $a \neq 1$ is an arbitrary constant.

If $a = 0$ and $\beta = \infty$, then by Theorems 3 and 4 we obtain the inequality

$$\left(\frac{|a-1|}{p}\right)^p \int^\infty t^{-a}|h|^p dt \leqslant \int_0^\infty t^{p-a}|\dot{h}|^p dt \tag{21}$$

which holds for $h \in H$; if $a > 1$, then $H = H_0$, if $a < 1$, then $H = H^0$. The equality in (21) holds only for $h \equiv 0$. Inequality (21) is well-known Hardy's integral inequality (cf. [8], Theorem 330, [7], [2]).

If $a = 0$ and $\beta < \infty$, then by Theorem 4 (i) for $a > 1$ and $h \not\equiv 0$ we obtain the inequality

$$\left(\frac{a-1}{p}\right)^p \int_0^\beta t^{-a}|h|^p dt + \left(\frac{a-1}{p}\right)^{p-1} \beta^{1-a}|h(\beta)|^p < \int_0^\beta t^{p-a}|\dot{h}|^p dt \tag{22}$$

which holds for $h \in H_0$. If $a > 0$ and $\beta = \infty$, then by Theorem 4 (ii) for $a < 1$ and $h \not\equiv 0$ we get the inequality

$$\left(\frac{1-a}{p}\right)^p \int_a^\infty t^{-a}|h|^p dt + \left(\frac{1-a}{p}\right)^{p-1} a^{1-a}|h(a)|^p < \int_a^\infty t^{p-a}|\dot{h}|^p dt \tag{23}$$

which holds for $h \in H^0$. Paper [11] is devoted to the derivation of inequalities (22) and (23).

In the interval $I = (0, \infty)$, by Theorems 2 and 3, one can obtain the inequalities with more general substitutions, i.e. $r = t^a$, $\varphi = t^k(1+t^b)^l$,

$s = \gamma t^a(1+t^b)^\beta$, where $a, b, k, l, \alpha, \beta$ and γ are some constants. The constants should be chosen so that the equation $(r|\dot\varphi|^{p-1}\operatorname{sgn}\dot\varphi)^\cdot + s\varphi^{p-1} = 0$ be satisfied. Those inequalities contain all inequalities of the same type which have been considered in [2]. Similarly, by Theorems 2 and 3, taking $I = (0, 1)$, $r = t^a$, $\varphi = t^k(1-t^b)^l$ and $s = \gamma t^a(1-t^b)^\beta$, where $a, b, k, l, \alpha, \beta$ and γ are some constants, we obtain the class of inequalities which contains the remaining inequalities of [2].

Example 2. Let $I = (0, 1)$, $r = 1$ and

$$t = \frac{p}{\pi}\sin\frac{\pi}{p}\int_0^\varphi (1-x^p)^{-1/p}dx.$$

Then by Theorems 2 and 3 we get the inequality

$$(p-1)\left(\frac{p}{\pi}\sin\frac{\pi}{p}\right)^{-p}\int_0^1 |h|^p dt \leqslant \int_0^1 |\dot h|^p dt \tag{24}$$

which holds for $h \in H_0$; it becomes the equality if and only if $h = c\varphi$, where $c = \text{const}$ (cf. [8], Theorem 256, and [2]).

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