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SOME REMARKS ON CONFORMALLY SYMMETRIC RIEMANNIAN SPACES

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According to Chaki and Gupta [1] an n-dimensional analytic Riemannian space with $n > 3$ is called *conformally symmetric* if the well-known Weyl's conformal tensor

$$C_{ijkl} = R_{ijkl} - aR_{jk}g_{il} + aR_{jl}g_{ik} - ag_{jk}R_{il} + ag_{jl}R_{ik} + bRg_{jk}g_{il} - bRg_{jl}g_{ik} \tag{1}$$

satisfies the condition

$$C_{ijkl,m} = 0, \tag{2}$$

where $a = 1/n-2$, $b = 1/(n-1)(n-2)$, and the comma in (2) indicates covariant differentiation with respect to the metric of the space.

Notation is derived from [2].

THEOREM 1. *Every conformally symetric space is conformally flat or the vector $R_{,i}$ is null.*

Proof. Differentiating (1) covariantly and taking into account (2) we get

$$R_{ijkl,m} - aR_{jk,m}g_{il} + aR_{jl,m}g_{ik} - ag_{jk}R_{il,m} + ag_{jl}R_{ik,m} + {} + bR_{,m}g_{jk}g_{il} - bR_{,m}g_{jl}g_{ik} = 0. \tag{3}$$

Since

$$R^r_{jkl,r} = R_{jk,l} - R_{jl,k} \quad \text{and} \quad R^r_{l,r} = \tfrac{1}{2}R_{,l},$$

we infer by contraction of (3) with g^{im} that

$$R_{jk,l} - R_{jl,k} = \frac{1}{2(n-1)}(g_{jk}R_{,l} - g_{jl}R_{,k}). \tag{4}$$

Now, differentiating (3) covariantly and using Ricci's identity, we obtain

$$R^s_{imp}T_{sjkl} - R^s_{jmp}T_{sikl} + R^s_{kmp}T_{slij} - R^s_{lmp}T_{skij} = 0, \tag{5}$$

where

(6) $$T_{sjkl} = R_{sjkl} - ag_{jk}R_{sl} + ag_{jl}R_{sk}.$$

The last relation, in view of (3) gives the equality

$$T_{sjkl,u} = aR_{jk,u}g_{sl} - aR_{jl,u}g_{sk} - bR_{,u}g_{jk}g_{sl} + bR_{,u}g_{jl}g_{sk},$$

whence

$$R^s_{imp}T_{sjkl,u} = aR_{limp}R_{jk,u} - aR_{kimp}R_{jl,u} - bR_{,u}g_{jk}R_{limp} + \\ + bR_{,u}g_{jl}R_{kimp}$$

and, consequently,

(7) $$R^s_{imp}T_{sjkl,u} - R^s_{jmp}T_{sikl,u} + R^s_{kmp}T_{slij,u} - R^s_{lmp}T_{skij,u} = 0.$$

On the other hand, in view of (7), the covariant differentiation of (5) gives

(8) $$R^s_{imp,u}T_{sjkl} - R^s_{jmp,u}T_{sikl} + R^s_{kmp,u}T_{slij} - R^s_{lmp,u}T_{skij} = 0,$$

Contracting (8) with g^{pu} and applying (4), we get

(9) $$(g_{mi}R_{,s}T^s_{jkl} - R_{,i}T_{mjkl}) - (g_{mj}R_{,s}T^s_{ikl} - R_{,j}T_{mikl}) + \\ + (g_{mk}R_{,s}T^s_{lij} - R_{,k}T_{mlij}) - (g_{ml}R_{,s}T^s_{kij} - R_{,l}T_{mkij}) = 0,$$

whence, by the subsequent contraction with g^{mi}, we come to

$$R_{,s}T^s_{jkl} = -aR_{lj}R_{,k} + aR_{kj}R_{,l} + bg_{jl}RR_{,k} - bg_{jk}RR_{,l}.$$

Hence

(10) $$g_{mi}R_{,s}T^s_{jkl} - R_{,i}T_{mjkl} = -ag_{mi}R_{lj}R_{,k} + ag_{mi}R_{kj}R_{,l} + \\ + bg_{mi}g_{jl}RR_{,k} + aR_{,i}g_{jk}R_{ml} - bRg_{mi}g_{jk}R_{,l} - aR_{,i}g_{jl}R_{mk} - R_{,i}R_{mjkl}.$$

However, it can be easily verified, by virtue of (9), (10) and (1), that

(11) $$R_{,i}C_{mjkl} - R_{,j}C_{mikl} + R_{,k}C_{mlij} - R_{,l}C_{mkij} = 0,$$

whence, by contraction with g^{mi} and in view of the equalities

$$C^s_{skl} = C^s_{jsl} = C^s_{jks} = 0,$$

it follows that

(12) $$R_{,s}C^s_{jkl} = 0.$$

Contracting now (11) with R^i and taking into consideration (12) and the well-known relations

$$C_{hijk} = C_{jkhi} = -C_{ihjk} = -C_{hikj},$$

we finally obtain

$$R_{,i}R_{,}{}^{i}C_{mjkl} = 0,$$

which completes the proof of Theorem 1.

As an immediate consequence of Theorem 1 we have the following

THEOREM 2. *Every conformally symmetric space with definite metric is conformally flat or its scalar curvature is constant.*

From Theorem 2 and (4) easily follows

THEOREM 3. *Every conformally symmetric space with definite metric is conformally flat or its Ricci tensor satisfies* $R_{ik,l} = R_{il,k}$.

REFERENCES

[1] M. C. Chaki and B. Gupta, *On conformally symmetric spaces*, Indian Journal of Mathematics 5 (1963), p. 113 - 122.

[2] K. Yano and S. Bochner, *Curvature and Betti numbers*, Princeton, 1953.

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