NUMBER OF ALGEBRAIC OPERATIONS IN IDEMPOTENT GROUPOIDS

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Let $\mathfrak{A}$ be an algebra. We say that an n-ary operation is *essentially n-ary* if it depends on every of its variables. We denote by $\omega_n = \omega_n(\mathfrak{A})$ the number of all essentially n-ary algebraic operations of $\mathfrak{A}$ (cf. Marczewski [2]). The aim of this paper is to find the minimal numbers ω_n for idempotent groupoids. Let us recall that for 2-dimensional proper diagonal algebras (cf. Płonka [4]) we have $\omega_2 = 2$ and $0 = \omega_3 = \omega_4 = \ldots$, and that for semilattices there is $\omega_n = 1$ for $n = 2, 3, \ldots$

The following shows that these groupoids are exceptional:

THEOREM. *For every idempotent groupoid G (with the exception of semilattices and the 2-dimensional proper diagonal algebras) we have $\omega_n(G) \geqslant n$ for $n \geqslant 3$.*

If $G = (G, \cdot)$ is an idempotent groupoid (and the operation $\cdot$ is essentially binary) we define the sequences of *simple* iterations of the operations $x \cdot y$ and $y \cdot x$ (cf. Marczewski [3]):

$$f_2(x_1, x_2) = x_1 \cdot x_2, \qquad f_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = f_2\big(f_n(x_1, x_2, \ldots, x_n), x_{n+1}\big),$$

$$g_2(x_1, x_2) = x_2 \cdot x_1, \qquad g_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = g_2\big(g_n(x_1, x_2, \ldots, x_n), x_{n+1}\big).$$

In the sequel f_n and g_n always denote these iterations.

LEMMA 1. *In every idempotent groupoid, if f_n is essentially n-ary, then f_{n+1} depends on the variables $x_3, x_4, \ldots, x_n$ and on one at least of the variables x_1, x_2 (an analogous statement is valid for g_{n+1}).*

Proof. The operation $x \cdot y$ is idempotent, so

$$f_{n+1}(x_2, x_2, x_3, \ldots, x_n, x_{n+1}) = f_n(x_2, x_3, \ldots, x_n, x_{n+1}).$$

This and the fact that f_n is essentially n-ary gives Lemma 1.

LEMMA 2. *If the operations f_3 and g_3 of an idempotent groupoid G are not essentially ternary, then G is a 2-dimensional proper diagonal algebra.*

Proof. Let us first remark that if f_3 (or g_3) is not essentially ternary, then the operation $x \cdot y$ is not commutative. Indeed, if it were commuta-

tive, Lemma 1 and the equality $f(x_1, x_2, x_3) = f(x_2, x_1, x_3)$ would result in contradiction with the assumption. Hence the operation $\cdot$ is not commutative and, according to Lemma 1, we have either

(1)
$$f_3(x_1, x_2, x_3) = x_2 \cdot x_3$$
or
(2)
$$f_3(x_1, x_2, x_3) = x_1 \cdot x_3 .$$

By applying (1) and Lemma 1 to the operation g_3 we get

$$x_1 \cdot x_2 = f_3(x_1, x_2, x_1 \cdot x_2) = g_3(x_2, x_1, x_2);$$

a contradiction, because there is either $x_1 \cdot x_2 = x_2 \cdot x_1$ or $x_1 \cdot x_2 = x_2$.

Analogous proof shows that equality (2) contradicts the equality $g_3(x_1, x_2, x_3) = x_3 \cdot x_2$, and so we come to the conclusion that G satisfies the equalities $x \cdot x = x$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot z$ of a 2-dimensional proper diagonal algebra (cf. Płonka [3]); thus Lemma 2 is proved.

By (φ) and (ψ) we denote propositions saying that all of the operations f_n (or g_n respectively) are essentially n-ary.

LEMMA 3. *Each idempotent non-diagonal groupoid G satisfies either (φ) or (ψ).*

Proof. Suppose that non-(φ) and non-(ψ). Then there exist the least integers n and m such that the operations f_n and g_m do not depend on all of their variables. According to Lemma 1 the following equalities are fulfilled in the groupoid G:

(3)
$$f_n(x_1, x_2, \ldots, x_n) = f_{n-1}(x_1, x_3, x_4, \ldots, x_{n-1}, x_n)$$
or
(4)
$$f_n(x_1, x_2, \ldots, x_n) = f_{n-1}(x_2, x_3, x_4, \ldots, x_{n-1}, x_n),$$
and
(5)
$$g_m(x_1, x_2, \ldots, x_m) = g_{m-1}(x_2, x_3, x_4, \ldots, x_{m-1}, x_m)$$
or
(6)
$$g_m(x_1, x_2, \ldots, x_m) = g_{m-1}(x_1, x_3, x_4, \ldots, x_{m-1}, x_m) .$$

Without loss of generality (because of Lemma 2 and of the assumption of this lemma) we can take $n \geqslant 4$.

Putting $x_n = f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ in (3) and (4) we get

(7)
$$f_{n-1}(x_1, x_2, \ldots, x_{n-1})$$
$$= g_3\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big),$$

(8)
$$f_{n-1}(x_1, x_2, \ldots, x_{n-1})$$
$$= g_3\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big) .$$

If we now multiply (7) on the left $m-3$ times by $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ and after every multiplication use (3), we get

$$(9) \quad f_{n-1}(x_1, x_2, \ldots, x_{n-1})$$

$$= g_m\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}),$$

$$\ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big).$$

If (6) is satisfied, then the right-hand side of the last equality is independent of x_2, and so (3) and (6) contradict each other; and if (5) and (9) are satisfied, then

$$f_{n-1}(x_1, x_2, \ldots, x_{n-1})$$

$$= g_m\big(x_{n-1}, f_{n-2}(x_1, \ldots, x_{n-2}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}),$$

$$f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big)$$

$$= g_m\big(x_{n-2}, f_{n-3}(x_1, x_2, \ldots, x_{n-3}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}),$$

$$f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big)$$

$$= \ldots = g_m\big(x_2, x_1, f_{n-2}(x_1, x_3, \ldots, x_{n-1}),$$

$$f_{n-2}(x_1, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big)$$

$$= g_{m-1}\big(x_1, f_{n-2}(x_1, x_3, \ldots, x_{n-1}), f_{n-2}(x_1, x_3, \ldots, x_{n-1}),$$

$$\ldots, f_{n-2}(x_1, x_3, \ldots, x_{n-1})\big).$$

This proves that $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ does not depend on x_2, and so (3) and (5) cannot be satisfied simultaneously. If we assume (4), we get from (8) in an analogous way the formula

$$(10) \quad f_{n-1}(x_1, x_2, \ldots, x_{n-1})$$

$$= g_m\big(x_{n-1}, f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$

$$f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, \ldots, x_{n-1})\big).$$

If (4) and (6) are satisfied, then we get a contradiction in view of (10), because $f_{n-1}(x_1, x_2, \ldots, x_{n-1})$ depends on x_1 and the right-hand side of (10) does not. If (5) and (10) are satisfied, then the right-hand side of (10) takes the form

$$g_{m-1}\big(f_{n-2}(x_1, x_2, \ldots, x_{n-2}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$

$$\ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$

$$= g_m\big(x_{n-2}, f_{n-3}(x_1, x_2, \ldots, x_{n-3}), f_{n-2}(x_2, x_3, \ldots, x_{n-1});$$

$$f_{n-2}(x_2, x_3, \ldots, x_{n-1}), \ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$

$$= \ldots = g_{m-1}\big(x_1 \cdot x_2, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$

$$\ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$

$$= g_m\big(x_2, x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$
$$\ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= g_{m-1}\big(x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}), f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$
$$\ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= f_{n-1}(x_1, x_2, \ldots, x_{n-1}) = g_m\big(x_1, f_{n-2}(x_2, x_3, \ldots, x_{n-1}),$$
$$\ldots, f_{n-2}(x_2, x_3, \ldots, x_{n-1})\big)$$
$$= f_{n-2}(x_2, x_3, \ldots, x_{n-1}).$$

So (5) and (4) cannot be satisfied simultaneously, because otherwise f_{n-1} would not depend on its first variable. Thus Lemma 3 has been proved.

LEMMA 4. *If an idempotent non-diagonal groupoid G satisfies non-(φ) (or non-(ψ), respectively), then $\omega_m(G) \geqslant m$ for $m = 2, 3, \ldots$*

Proof. Suppose that non-(φ) (the proof is analogous if we suppose that non-(ψ)). Then either (3) or (4) is satisfied and we conclude from Lemma 3 that $g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1})$, where $m = 3, 4, \ldots$ and $k = 2, 3, \ldots, m$, are essentially m-ary. Suppose that for different integers k and l we have

(11) $g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1})$
$$= g_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}).$$

Using (3) and (11) we get

$$f_{n-1}\big(g_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_n\big(x_{k-1}, g_{m-1}(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-2}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_{n-1}(x_{k-1}, y_2, y_3, \ldots, y_{n-1})$$
$$= f_{n-1}\big(g_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_n\big(x_{l-1}, g_{m-1}(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-2}), y_2, y_3, \ldots, y_{n-1}\big)$$
$$= f_{n-1}(x_{l-1}, y_2, y_3, \ldots, y_{n-1}).$$

It means that f_{n-1} does not depend on its first variable. Using analogously (11) and (4) we come to a contradiction with the assumption on the operation f_{n-1}. So we have $\omega_m \geqslant m$ for $m = 2, 3, \ldots$

LEMMA 5. *If $f(x_1, x_2, \ldots, x_n)$ is an arbitrary essentially n-ary algebraic operation of the idempotent groupoid G which satisfies (φ) or (ψ), and if $f(x \cdot y, x_2, x_3, \ldots, x_n)$ does not depend on all of its variables, then $\omega_m(G) \geqslant m$ for $m = 2, 3, \ldots$*

Proof. In virtue of the assumption on the operations f and $f(x \cdot y, x_2, x_3, \ldots, x_n)$ one of the following equalities holds in G:

(12) $f(x \cdot y, x_2, x_3, \ldots, x_n) = f(y, x_2, x_3, \ldots, x_n),$

(13) $f(x \cdot y, x_2, x_3, \ldots, x_n) = f(x, x_2, x_3, \ldots, x_n).$

Suppose that (φ) is satisfied (the proof is analogous if we suppose that (ψ) is satisfied), and suppose that for different k and l and for some m we have

(14) $\qquad f_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1})$

$$= f_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}).$$

Then

$$f\big(f_m(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-1}), y_2, y_3, \ldots, y_n\big)$$

$$= f\big(f_{m-1}(x_k, x_{k+1}, \ldots, x_m, x_1, x_2, \ldots, x_{k-2}) \cdot x_{k-1}, y_2, y_3, \ldots, y_n\big)$$

$$= f\big(f_m(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-1}), y_2, y_3, \ldots, y_n\big)$$

$$= f\big(f_{m-1}(x_l, x_{l+1}, \ldots, x_m, x_1, x_2, \ldots, x_{l-2}) \cdot x_{l-1}, y_2, y_3, \ldots, y_n\big).$$

If (12) or (13) holds, we conclude from the last equality that f does not depend on its first variable, what obviously contradicts the assumption. Hence (14) does not hold for $k \neq l$, and so $\omega_m(G) \geqslant m$.

LEMMA 6. *If f is an arbitrary essentially n-ary algebraic operation of the idempotent groupoid G which satisfies (φ) and for a k_0 we have $\omega_{k_0}(G)$ $< k_0$, then the operation*

$$f\big(f_m(x_1, x_2, \ldots, x_m), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables; if G satisfies (ψ), then the operation

$$f\big(g_m(x_1, x_2, \ldots, x_m), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables.

We give a proof by induction on m in the case in which G satisfies (φ). If $m = 2$, the thesis follows from Lemma 5. Suppose that the thesis of the lemma holds for a fixed $m \geqslant 2$. Then, according to the inductive assumption and Lemma 5, the operation

$$f\big(g_{m+1}(x_1, x_2, \ldots, x_{m+1}), y_2, y_3, \ldots, y_n\big)$$

$$= f\big(g_m(x_2 \cdot x_1, x_3, \ldots, x_{m+1}), y_2, y_3, \ldots, y_n\big)$$

depends on all of its variables. Thus the lemma is proved.

Let us now introduce a definition: the operation $f(x_1, x_2, \ldots, x_n)$ admits the permutation $\sigma \epsilon S_n$ if

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

The permutation

$$\begin{pmatrix} 1 & 2 & \ldots & n-k+1 & n-k+2 & \ldots & n \\ k & k+1 & \ldots & n & 1 & & k-1 \end{pmatrix}$$

will be denoted by σ_n^k $(k = 2, 3, \ldots, n)$.

LEMMA 7. *If the idempotent groupoid is non-abelian, the operation* f_p *(and* g_p, *analogously) does not admit any permutation* σ_p^k *for any prime* p.

Proof. Let $p \geqslant 5$ and suppose that for a k_0 the operation f_p admits $\sigma_p^{k_0}$. Then f_p admits every permutation σ_p^k, because p is prime.

First we prove that the binary operation

$$f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y)$$

is commutative. Indeed,

$$f_{p+1}(x, x, \ldots, x, y, y, \ldots, y) = f_p(x, x, \ldots, x, y, y, \ldots, y) \cdot y$$

$$= f_{p-1}(y, y, \ldots, y, x, x, \ldots, x) \cdot y = f_p(y, y, \ldots, y, x, x, \ldots, x, y)$$

$$= f_p(y, y, \ldots, y, x, x, \ldots, x) = f_{p+1}(y, y, \ldots, y, \underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}),$$

whence

$$x \cdot y = f_p(x, x, \ldots, x, y) = f_p(y, x, x, \ldots, x)$$

$$= f_{(p-1)/2}\left(f_{p+1}(\underbrace{y, y, \ldots, y}_{(p+1)/2 \text{ times}}, x, x, \ldots, x), \underbrace{x, x, \ldots, x}_{(p-3)/2 \text{ times}}\right)$$

$$= f_{(p-1)/2}\left(f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y), x, x, \ldots, x\right)$$

$$= f_{(p-1)/2}\left(f_{(p+3)/2}(x, x, \underbrace{y, y, \ldots, y}_{(p+1)/2 \text{ times}}), x, x, \ldots, x\right)$$

$$= f_p(x, y, y, \ldots, y, x, x, \ldots, x) = f_p(x, x, \ldots, x, y, y, \ldots, y)$$

$$= f_{p+1}(\underbrace{x, x, \ldots, x}_{(p+1)/2 \text{ times}}, y, y, \ldots, y).$$

It means that the operation $x \cdot y$ is commutative, which contradicts the assumption.

For $p = 2$ the lemma is obvious, and if $p = 3$ and f_3 admits the permutation σ_3^k, then the following equalities hold:

$$f_3(x_1, x_2, x_3) = f_3(x_2, x_3, x_1) = f_3(x_3, x_1, x_2).$$

Putting $x_3 = x_1 \cdot x_2$ in the last equality, we have

$$x_1 \cdot x_2 = f_3(x_1 \cdot x_2, x_1, x_2) = f_3(x_1, x_2, x_1) \cdot x_2 = f_3(x_1, x_1, x_2) \cdot x_2$$

$$= f_3(x_1, x_2, x_3) = f_3(x_2, x_2, x_1) = x_2 \cdot x_1,$$

a contradiction.

LEMMA 8. *If*

$$f = f(x_1, x_2, \ldots, x_n) \quad and \quad g = g(y_1, y_2, \ldots, y_m)$$

are idempotent operations, if f does not admit any permutation σ_n^i and g does not admit any permutation σ_m^j, then the operation

$$\hat{f}(g)(x_1, x_2, \ldots, x_{n \cdot m})$$

$$= f\big(g(x_1, x_2, \ldots, x_m), g(x_{m+1}, x_{m+2}, \ldots, x_{2m}), \ldots, g(x_{(n-1)m+1}, \ldots, x_{n \cdot m})\big)$$

does not admit any permutation $\sigma_{n \cdot m}^k$.

Proof. Suppose that $\hat{f}(g)$ admits the permutation $\sigma_{n \cdot m}^k$ and let $k = (s-1)m+1$, where $s = 2, 3, \ldots, n$. Then

$$f(x_1, x_2, \ldots, x_n)$$

$$= \hat{f}(g)(x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_n, x_n, \ldots, x_n)$$

$$= \hat{f}(g)(x_s, x_s, \ldots, x_s, x_{s+1}, x_{s+1}, \ldots, x_{s+1}, \ldots, x_n, x_n,$$

$$\ldots, x_n, x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_{s-1}, x_{s-1}, \ldots, x_{s-1})$$

$$= f(x_s, x_{s+1}, \ldots, x_n, x_1, x_2, \ldots, x_{s-1}),$$

which contradicts the assumption of the lemma. And if $k \neq (s-1)m+1$ for $s = 2, 3, \ldots, n$, then there exists an r such that $2 \leqslant r \leqslant m$ and

$$g(x_1, x_2, \ldots, x_m)$$

$$= \hat{f}(g)(x_1, x_2, \ldots, x_m, x_1, x_2, \ldots, x_m, \ldots, x_1, x_2, \ldots, x_m)$$

$$= \hat{f}(g)(x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1}, x_r, x_{r+1},$$

$$\ldots, x_m, x_1, x_2, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1})$$

$$= g(x_r, x_{r+1}, \ldots, x_m, x_1, x_2, \ldots, x_{r-1})$$

which contradicts the assumption on the operation g. Thus Lemma 8 is proved.

Now for each prime p we define inductively some sequences of algebraic operations of the groupoid G; namely,

$$F_1^{(p)}(x_1, x_2, \ldots, x_p) = f_p(x_1, x_2, \ldots, x_p),$$

$$F_{n+1}^{(p)}(x_1, x_2, \ldots, x_{p^{n+1}}) = \hat{F}_n^{(p)}(f_p)(x_1, x_2, \ldots, x_{p^{n+1}}).$$

Analogously, we define the sequence $G_n^{(p)}$ starting with g_p.

LEMMA 9. *If an idempotent groupoid G satisfies (φ) and $\omega_{k_0}(G) < k_0$ for a k_0, then for every p and n the operations $F_n^{(p)}$ depend on all their variables (analogous proposition holds for $G_n^{(p)}$ if (ψ)).*

Proof is by induction on n and uses Lemma 6.

LEMMA 10. *For every idempotent non-abelian groupoid the operations $F_n^{(p)}$ and $G_n^{(p)}$ do not admit any permutation σ_{pn}^k.*

Proof immediately follows from Lemmas 7 and 8.

Now we can prove the theorem in the case of non-abelian (idempotent and non-diagonal) groupoid. Suppose that we have $\omega_{k_0}(G) < k_0$ for a $k_0 = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ ($p_1, p_2, \dots, p_k$ are primes). According to Lemmas 3 and 4 we can assume that the groupoid G satisfies (φ) and (ψ), and from Lemmas 9 and 10 we conclude that $F_{n_j}^{(p_j)}$ ($j = 1, 2, \dots, k$) depend on all their variables and do not admit any permutation of the type σ_n^i. Then, in view of Lemmas 6 and 8, the operations

$$H_1 = F_{n_1}^{(p_1)}, \qquad H_2 = \hat{H}_1(F_{n_2}^{(p_2)}), \dots, H_k = \hat{H}_{k-1}(F_{n_k}^{(p_k)})$$

depend on all their variables and also do not admit any permutation of the type σ_n^i; however, the operation H_k is essentially k_0-ary and does not admit any permutation $\sigma_{k_0}^i$, whence $\omega_{k_0}(G) \geqslant k_0$, which ends the proof of the theorem in this case.

Now we take the case of an abelian groupoid.

LEMMA 11. *If G is an idempotent abelian groupoid, then, for every n, $f_n = g_n$ and f_n is essentially n-ary.*

Proof. For $n = 2$ the lemma is obvious. Idempotency and commutativity of the operation $\cdot$ and Lemma 1 imply that if f_n is essentially n-ary, then f_{n+1} is essentially $(n+1)$-ary.

LEMMA 12. *If G is an idempotent abelian non-associative groupoid, then for $n \geqslant 3$ the operation f_n does not admit any permutation $\varrho_k = (k, k+1)$ of the set $\{1, 2, \dots, n\}$, where $2 \leqslant k \leqslant n-1$.*

Proof. The operation $x \cdot y$ is not associative and hence we have the thesis in case $n = 3$. Suppose that f_n does not admit any permutation ϱ_k ($2 \leqslant k \leqslant n-1$) and let

$$f_{n+1}(x_1, x_2, \dots, x_n, x_{n+1}) = f_n(x_1, x_2, \dots, x_n) \cdot x_{n+1}$$

admit the permutation ϱ_k for $3 \leqslant k \leqslant n$. Then, putting $x_1 = x_2$ in the last equality, we get the contradiction with the inductive assumption. And if

$$f_{n+1}(x_1, x_2, \dots, x_n, x_{n+1}) = f_{n+1}(x_1, x_3, x_2, x_4, \dots, x_n, x_{n+1}),$$

then for $x_{n+1} = f_n(x_1, x_2, \dots, x_{n-1}, x_n)$ we have

$$f_n(x_1, x_2, \dots, x_n) = f_n(x_1, x_3, x_2, x_4, \dots, x_n) \cdot f_n(x_1, x_2, x_3, x_4, \dots, x_n)$$

$$= f_n(x_1, x_2, x_3, x_4, \dots, x_n) \cdot f_n(x_1, x_3, x_2, x_4, \dots, x_n)$$

$$= f_n(x_1, x_3, x_2, x_4, \dots, x_n) \cdot f_n(x_1, x_3, x_2, x_4, \dots, x_n)$$

$$= f_n(x_1, x_3, x_2, x_4, \dots, x_n).$$

This proves that f_n admits the permutation ϱ_2, which contradicts the inductive assumption.

LEMMA 13. *For every idempotent abelian non-associative groupoid G the operations f_n do not admit any of the permutations σ_n^k.*

Proof. In fact, if for $2 \leqslant k \leqslant n-1$ the operation f_n admitted the permutation σ_n^k, then, the operation $x \cdot y$ being commutative, the operation f_n would admit the permutation ϱ_k, which contradicts Lemma 12. And if

$$f_n(x_1, x_2, \ldots, x_n) = f_n(x_n, x_1, x_2, \ldots, x_{n-1}),$$

f_n would admit ϱ_2, which also contradicts Lemma 12.

If the abelian idempotent groupoid is not a semilattice, the theorem follows from Lemmas 11 and 13.

Let us remark that the equality in the thesis of the theorem cannot be improved as there exist groupoids for which $\omega_n = n$ (Płonka [5]).

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