COLLOQUIUM MATHEMATICUM

VOL. XXXII 1975 FASC. 2

# *METRIC CONVEXITY AND NORMABILITY OF METRIC LINEAR SPACES*

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**1. Introduction.** Let $X$ be a first countable linear topological space over the real numbers $R$ or the complex numbers $C$. Then (see [5], p. 28) there exists a function $p\colon X \to [0, \infty)$ such that, for all $x, y \in X$ and scalars $\lambda$,

(P1) $p(x) = 0$ if and only if $x = 0$;

(P2) $p(\lambda x) \leqslant p(x)$ if $|\lambda| \leqslant 1$;

(P3) $p(x+y) \leqslant p(x)+p(y)$;

(P4) the topology of $X$ is given by the metric $d(x, y) = p(x-y)$.

If $X$ is a linear space and $p\colon X \to [0, \infty)$ satisfies (P1)-(P3), then $p$ will be called a *pseudonorm*, and the pair $(X, p)$ a *pseudonormed space*. If $X$ is a linear topological space and $p\colon X \to [0, \infty)$ satisfies (P1)-(P4), then the pair $(X, p)$ will be called a *metric linear space*. A *norm* is a pseudonorm which is absolutely homogeneous.

Following Menger [4], a subset $A$ of a metric space $(M, d)$ will be called *metric convex* if, for each $x, y \in A$, $x \neq y$, there exists $z \in A$ such that $d(x, z) = d(z, y) = \frac{1}{2}d(x, y)$. If $M$ itself is metric convex, then we shall usually say that the metric $d$ is *metric convex*. In [2], p. 41, it is shown that, for a complete space $(M, d)$, there is a weaker condition equivalent to the convexity of $d$. If $x$ and $y$ are distinct points of $M$, then $z \in M$ is *metrically between* $x$ and $y$ provided $z \neq x$, $z \neq y$, and $d(x, y) = d(x, z)+d(z, y)$. If each pair of distinct points in a complete metric space $(M, d)$ has a metrically between point, then, given $x, y \in M$, there is an isometry $f$ mapping the interval $[0, d(x, y)]$ into $M$ so that $f(0) = x$ and $f\big(d(x, y)\big) = y$. In particular, $d$ is metric convex. The image under $f$ of $[0, d(x, y)]$ is called a *metric segment* between $x$ and $y$.

There is an equivalent formulation of metric convexity in pseudonormed spaces; namely, a pseudonorm $p$ is metric convex if and only if, for each $x \in X$, there exists $y \in X$ such that $p(y) = p(x-y) = \frac{1}{2}p(x)$. If each such $y$ is unique, then $p$ will be called *strictly metric convex*. It is easy to verify that when $p$ is a norm, this definition coincides with the usual definition of a strictly convex norm.

If $A$ is a subset of a linear topological space $X$, then $\mathrm{co}(A)$ and $\overline{\mathrm{co}}(A)$ denote the convex hull and closed convex hull of $A$, respectively. If $K$ is a convex set, then $\partial K$ denotes the set of extreme points of $K$. For a pseudonorm $p$, the closed $p$-sphere centered at $x$ with radius $\varepsilon \geqslant 0$ will be denoted by $B(x, \varepsilon)$.

In Section 2 conditions are investigated under which the space is normable. A preliminary result shows that a continuous strictly metric convex pseudonorm is a norm. A subsequent characterization of those metric convex pseudonorms which are strictly metric convex is obtained. It is also shown that, for finite-dimensional spaces, metric convexity characterizes those pseudonorms which are norms. The final result shows that a locally convex metric linear space with metric convex pseudonorm is necessarily normable.

In Section 3 examples are presented which show that the above results are not true in more general settings.

**2. Main results.** The following two preliminary lemmas will be useful in the sequel.

LEMMA 1. *Let $p$ be a continuous pseudonorm on a linear topological space $X$. Then $p$ is a norm if and only if $p(\frac{1}{2}x) \leqslant \frac{1}{2}p(x)$ for each $x \in X$.*

Proof. The necessity is immediate from the definition of a norm. Suppose $p(\frac{1}{2}x) \leqslant \frac{1}{2}p(x)$ for each $x \in X$. Then (P3) implies $p(\frac{1}{2}x) = \frac{1}{2}p(x)$, and a simple induction argument shows that $p(2^k x) = 2^k p(x)$ for each integer $k$ and each $x \in X$. By (P3), it is sufficient to show (for either $R$ or $C$) that $p(\lambda x) \leqslant \lambda p(x)$ for $\lambda \in [0, 1]$. But if $n/2^k$ is a dyadic rational in $[0, 1]$, then, from the above and (P3),

$$p\left(\frac{n}{2^k}x\right) = \frac{1}{2^k}p(nx) \leqslant \frac{n}{2^k}p(x),$$

and the result follows from the continuity of $p$.

LEMMA 2. *Let $(X, p)$ be a metric linear space. Then $p$ is a norm if and only if $p$ is metric convex and the spheres $B(x, \varepsilon)$ are convex.*

Proof. The necessity follows easily from the definition of a norm. If $p$ is metric convex and the spheres are convex, then, for each $x \in X$, the set

$$C_x = B\big(0, \tfrac{1}{2}p(x)\big) \cap B\big(x, \tfrac{1}{2}p(x)\big)$$

is convex and non-empty. If $y \in C_x$, then, clearly, $x - y \in C_x$. Thus

$$\tfrac{1}{2}x = \tfrac{1}{2}y + \tfrac{1}{2}(x-y) \in C_x,$$

and it follows that $p(\frac{1}{2}x) \leqslant \frac{1}{2}p(x)$. By Lemma 1, $p$ is a norm.

THEOREM 1. *Let $p$ be a continuous strictly metric convex pseudonorm on a linear topological space $X$. Then $p$ is a strictly convex norm.*

Proof. Let $x \in X$ and let $y$ be an element of the set $C_x$ defined above. Then $x-y \in C_x$ and, since $p$ is strictly metric convex, $y = x-y$. Therefore, $p(\frac{1}{2}x) \leqslant \frac{1}{2}p(x)$ and, by Lemma 1, $p$ is a norm.

THEOREM 2. *Let $(X, p)$ be a complete metric linear space and let $p$ be metric convex. Then $p$ is a strictly convex norm if and only if each closed metric convex subset of $X$ is convex.*

Proof. Suppose $p$ is a strictly convex norm and let $A$ be a closed metric convex subset of $X$. Since $X$ is complete and $A$ is closed, it is enough to show that $\frac{1}{2}(x+y) \in A$ whenever $x, y \in A$. Since $A$ is metric convex, there exists a $z$ in $A$ such that

$$p(z-x) = p(y-z) = \tfrac{1}{2}p(x-y).$$

Hence, $p$ being a strictly convex norm, we get $z = \frac{1}{2}(x+y)$.

Suppose each closed metric convex subset of $X$ is convex and let $x \in X$. Let $S$ be a metric segment between 0 and $x$. Then $S$ is closed and metric convex. Thus $S = \{\lambda x\colon 0 \leqslant \lambda \leqslant 1\}$ and there exists $\lambda \in (0, 1)$ such that

$$p(\lambda x) = p\big((1-\lambda)x\big) = \tfrac{1}{2}p(x).$$

For either of the cases $\lambda \leqslant \frac{1}{2}$ or $1-\lambda \leqslant \frac{1}{2}$ we have $p(\frac{1}{2}x) \leqslant \frac{1}{2}p(x)$. Hence, by Lemma 1, $p$ is a norm. Suppose that $p(x) = p(y) = 1$ and $p(x+y) = 2$. Then, clearly, $p(x+y) = p(x)+p(y)$. Since

$$p\big(x-(x-y)\big) = p(y) = \tfrac{1}{2}p\big(x-(-y)\big)$$

and

$$p\big(x-y-(-y)\big) = p(x) = \tfrac{1}{2}p\big(x-(-y)\big),$$

it follows that $x-y = \frac{1}{2}x+\frac{1}{2}(-y)$. Hence $x = y$, and $p$ is a strictly convex norm.

The special case of Theorem 2 for finite-dimensional $(X, p)$ with the Euclidean norm $p$ appears in [2], p. 43. A related result appears in [6], p. 101.

The argument used in the proof of the sufficiency of Theorem 2 shows that every metric convex pseudonorm on a one-dimensional space is a norm. We have the following extension to spaces of finite dimension.

THEOREM 3. *Let $(X, p)$ be a finite-dimensional metric linear space. Then $p$ is a norm if and only if $p$ is metric convex.*

Proof. The necessity is immediate. To prove the sufficiency let us assume that $p$ is metric convex. By Lemma 2, it is enough to show that the spheres $B(0, \varepsilon)$ are convex. We first show that they are compact. For a sufficiently small $\varepsilon > 0$, this is a consequence of the local compactness of the space. If $x \in B(0, 2\varepsilon)$, then there exists $y \in X$ such that $y$ and $x-y$ are in $B(0, \varepsilon)$. Hence $B(0, 2\varepsilon) = B(0, \varepsilon) + B(0, \varepsilon)$. By induction, it follows that, for each $\varepsilon > 0$ and for each non-negative integer $n$,

$$B(0, 2^n\varepsilon) = \underbrace{B(0, \varepsilon) + \ldots + B(0, \varepsilon)}_{2^n \text{ summands}}.$$

Since the sum of compact sets is compact, each sphere $B(0, \varepsilon)$ is compact.

For each $\varepsilon > 0$, let $K(\varepsilon) = \mathrm{co}(B(0, \varepsilon))$. Then $K(\varepsilon)$ is compact and convex, so that, by Milman's theorem (see [4], p. 68), $\partial K(\varepsilon) \subseteq B(0, \varepsilon)$. We will show by induction that if $x_0 \in \partial K(\varepsilon)$, then

$$p\left(\frac{1}{2^n} x_0\right) = \frac{1}{2^n} p(x_0)$$

for each non-negative integer $n$. The assertion is trivial for $n = 0$. Suppose it is true for $n = k-1$. Then there exists $z \in X$ such that

$$p(z) = p\left(\frac{1}{2^{k-1}} x_0 - z\right) = \frac{1}{2} p\left(\frac{1}{2^{k-1}} x_0\right) = \frac{1}{2^k} p(x_0) \leqslant \frac{\varepsilon}{2^k}.$$

Thus $2^k z$ and $2x_0 - 2^k z$ are in $B(0, \varepsilon) \subseteq K(\varepsilon)$. Since $x_0$ is an extreme point of $K(\varepsilon)$ and $x_0 = \frac{1}{2}(2^k z) + \frac{1}{2}(2x_0 - 2^k z)$, we have $2^k z = 2x_0 - 2^k z$. Therefore, $z = x_0/2^k$ and the assertion is true for $n = k$. As in the proof of Lemma 1, we have $p(\lambda x_0) \leqslant \lambda p(x_0)$ for all $\lambda \in [0, 1]$. Let

$$y = \sum_{i=1}^{n} \lambda_i x_i, \quad \text{where } \lambda_i \geqslant 0, \ \sum_{i=1}^{n} \lambda_i = 1 \text{ and } x_i \in \partial K(\varepsilon).$$

Then

$$p(y) \leqslant \sum_{i=1}^{n} p(\lambda_i x_i) \leqslant \sum_{i=1}^{n} \lambda_i p(x_i) \leqslant \sum_{i=1}^{n} \lambda_i \varepsilon = \varepsilon.$$

Hence, by the Krein-Milman theorem, $K(\varepsilon)$ is a subset of $B(0, \varepsilon)$, and so $B(0, \varepsilon)$ is convex.

By imposing an additional condition on the space $X$, we obtain the following result:

THEOREM 4. *Let $(X, p)$ be a locally convex metric linear space with a metric convex pseudonorm $p$. Then $p$ is normable.*

Proof. By a well-known result of Kolmogorov (see, e.g., [5], p. 41), it is sufficient to show the existence of a convex bounded neighborhood of 0. Since

$$B\left(0, \frac{1}{2^{n+1}}\right) + B\left(0, \frac{1}{2^{n+1}}\right) = B\left(0, \frac{1}{2^n}\right) \quad \text{for each } n,$$

we have, as in the proof of Theorem 3,

$$\underbrace{B\left(0, \frac{1}{2^{n+k}}\right) + \ldots + B\left(0, \frac{1}{2^{n+k}}\right)}_{2^k \text{ summands}} = B\left(0, \frac{1}{2^n}\right) \quad \text{for each } n \text{ and } k.$$

Hence, putting $K_n = \mathrm{co}(B(0, 1/2^n))$, we obtain

$$\underbrace{K_{n+k} + \ldots + K_{n+k}}_{2^k \text{ summands}} = 2^k K_{n+k} = K_n.$$

Thus, since the sets $K_n$ form a base at 0, they are bounded. In particular, $K_0$ is bounded.

We note that Theorems 3 and 4 hold for almost convex pseudonorms (see [1]).

**3. Examples.** In example (i) below we present a metric linear space which is not locally convex but has a metric convex pseudonorm. This example shows that Theorem 3 is not true for infinite-dimensional spaces. It also shows that the hypothesis of local convexity cannot be deleted from Theorem 4. In example (ii) below we present a locally convex metric linear space with a convex pseudonorm which is not a norm.

(i) Let $L^{1/2}[0, 1]$ denote the space of Lebesgue measurable scalar functions $f$ on $[0, 1]$ such that

$$\int_0^1 |f|^{1/2} < \infty.$$

Define $p$ on $L^{1/2}[0, 1]$ by

$$p(f) = \int_0^1 |f|^{1/2}.$$

Then $p$ is, clearly, a pseudonorm which is not a norm. Furthermore, the space $(L^{1/2}[0,1], p)$ is not locally convex (see [5], p. 34) and, therefore, not normable. We now show that $p$ is metric convex. Let $f \in L^{1/2}[0, 1]$ and let

$$F(x) = \int_0^x |f|^{1/2}.$$

Then, by [3], p. 272, there exists $t \in [0, 1]$ such that

$$F(t) = \int_0^t |f|^{1/2} = \tfrac{1}{2}\int_0^1 |f|^{1/2}.$$

Let $g \in L^{1/2}[0,1]$ be the characteristic function of the subset $[0,t]$ of $[0,1]$. Then

$$p(fg) = \int_0^1 |fg|^{1/2} = \int_0^t |f|^{1/2} = \tfrac{1}{2}\int_0^1 |f|^{1/2}$$

and

$$p(f-fg) = \int_0^1 |f-fg|^{1/2} = \int_0^1 |f(1-g)|^{1/2} = \int_t^1 |f|^{1/2} = \tfrac{1}{2}\int_0^1 |f|^{1/2}.$$

Hence $p(fg) = p(f-fg) = \frac{1}{2}p(f)$, and so $p$ is a metric convex pseudonorm.

(ii) Let $L^1[0,1]$ be the space of Lebesgue integrable scalar functions on $[0,1]$, with its usual norm

$$\|f\| = \int_0^1 |f|.$$

Define $p$ on $L^1[0,1]$ by

$$p(f) = \|f\| + \int_0^1 |f|^{1/2}.$$

Then $p$ is, clearly, a pseudonorm on $L^1[0,1]$ which is not a norm. Furthermore, by Hölder's inequality for $0 < p < 1$ (see [3], p. 191), we have

$$\|f\| \leqslant p(f) \leqslant \|f\| + \|f\|^{1/2} \quad \text{for each } f \in L^1[0,1],$$

and so $p$ is topologically equivalent to $\|\cdot\|$. We now show that $p$ is metric convex. Let $f \in L^1[0,1]$ and let

$$F(x) = \int_0^x (|f| + |f|^{1/2}).$$

Then, by [3], p. 272, there exists $t \in [0,1]$ such that

$$F(t) = \int_0^t (|f| + |f|^{1/2}) = \tfrac{1}{2}F(1) = \tfrac{1}{2}p(f).$$

Let $g \in L^1[0,1]$ be defined by $g = f$ on $[0,t]$ and by $g = 0$ in $(t,1]$. Then

$$p(g) = \int_0^1 |g| + \int_0^1 |g|^{1/2} = \int_0^t |f| + \int_0^t |f|^{1/2} = \tfrac{1}{2}p(f).$$

Similarly, we have $p(f-g) = \frac{1}{2}p(f)$. Thus $p$ is metric convex.

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*Reçu par la Rédaction le 5. 8. 1972*