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ON SOME FORM DETERMINED BY A DISTRIBUTION ON A RIEMANNIAN MANIFOLD

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In the present paper we consider the orientation form of a distribution E defined on a Riemannian manifold M. It is proved that, for E integrable, that form is closed if and only if leaves of E are minimal and the orthogonal complement $E^{\perp}$ is involutive.

Relations between the orientation form and the property of having totally geodesic leaves are also studied.

Throughout the paper, M denotes an m-dimensional connected Riemannian manifold of class C^{∞} with a Riemannian metric g and the Levi-Cività connection ∇. Let $E \subset TM$ denote an arbitrary but fixed (smooth) n-dimensional distribution over M, and let $E^{\perp}$ be the orthogonal complement of E.

The distribution E determines a 1-dimensional subbundle $E^{*} \subset \Lambda^{n}(T^{*}M)$ defined as follows:

$\varphi \in E^{*} \cap \Lambda^{n}(T_{p}^{*}M)$ iff, for any $Y \in E^{\perp} \cap T_{p}M$,

$$\varphi(Y, \cdot, \ldots, \cdot) = 0 \quad \text{for } p \in M.$$

The natural metric in E^{*} gives rise to the subspace E_{0} of normed n-covectors:

$\varphi \in E_{0} \cap \Lambda^{n}(T_{p}^{*}M)$ iff $\varphi \in E^{*}$ and $|\varphi(X_{1}, \ldots, X_{n})| = 1$ for some (any) orthonormal frame $(X_{1}, \ldots, X_{n})$ in $E \cap T_{p}M$, $p \in M$.

E_{0} is a (not necessarily connected) 2-fold smooth covering of M.

Definition 1. The distribution E is *orientable* if E_{0} is not connected.

Remark 1. It is easy to see that two different Riemannian metrics give rise to diffeomorphic coverings of M. Hence the notion of orientability depends on the distribution only.

Definition 2. An *orientation form* of E with respect to the Riemannian metric g over an open subset U of M is any continuous section $\omega \in \Gamma(U, E_{0})$.

The fact that E_{0} is a smooth covering of M implies smoothness of the orientation form.

Remark 2. We can always define the orientation form locally. A distribution admits a globally defined orientation form if and only if it is orientable. One can define the orientability of E in a different way by considering the action of the fundamental group $\pi(M, p_0)$ (p_0 fixed in M) over the 2-element set of orientations of the space $E\cap T_{p_0}M$. The action defines an induced homomorphism $\pi(M, p_0)\to\mathfrak{S}_2$ (the permutation group of a 2-element set). The definition is as follows: E is *orientable* if the induced homomorphism $\pi(M, p_0)\to\mathfrak{S}_2$ is trivial.

Definition 3. The distribution E is *minimal* at a point $p\in M$ if for some (any) orthonormal frame $(X_1, \ldots, X_n)$ of E over a neighbourhood of p we have

$$\Big(\sum_i \nabla_{X_i} X_i\Big)^{\perp}(p) = 0,$$

where $^{\perp}$ denotes the component normal to E.

Definition 4. A distribution F is *involutive* at a point q if for any smooth $Z_1, Z_2\in\Gamma(F)$ we have

$$[Z_1, Z_2](q)\in F_q.$$

THEOREM 1. *Suppose that the distribution $E\subset TM$ admits the orientation form ω over an open set $U\subset M$. Then for $p\in U$ the following two conditions are equivalent*:

(i) $d\omega(p) = 0$;

(ii) *E is minimal at p and $E^{\perp}$ is involutive at p.*

Proof. Let $V\subset U$ be a connected neighbourhood of p such that there exists a basis of E over V. Let $(X_1, \ldots, X_n)$ be a fixed orthonormal frame of E over V, X_i being vector fields on V. Changing, if necessary, the order of the n-tuple we may assume that

$$\omega(X_1, \ldots, X_n) = 1.$$

This and the condition $\omega\in\Gamma(U, E^*)$ make it possible to compute the value of $\omega(X_1, \ldots, X_{i-1}, Z, X_{i+1}, \ldots, X_n)$ for any $Z\in\mathfrak{X}(M)$ and $i\leqslant n$ by the formula

$$\omega(X_1, \ldots, X_{i-1}, Z, X_{i+1}, \ldots, X_n) = g(Z, X_i).$$

Let Y, Y_1, Y_2 be arbitrary vector fields orthogonal to E. Condition (i) is equivalent to the conjunction

$$d\omega(Y, X_1, \ldots, X_n)(p) = 0$$

and

$$d\omega(Y_1, Y_2, X_1, \ldots, \hat{X}_i, \ldots, X_n)(p) = 0 \quad \text{for } i\leqslant n,$$

where $\hat{}$ means that the term is omitted.

We obtain successively

$$\begin{aligned} & d\omega(Y, X_1, \ldots, X_n) \\ & = Y\big(\omega(X_1, \ldots, X_n)\big) + \sum_i (-1)^i \omega([Y, X_i], X_1, \ldots, \hat{X}_i, \ldots, X_n) \\ & = -\sum_i g([Y, X_i], X_i) \\ & = -\sum_i g(\nabla_Y X_i, X_i) + \sum_i g(\nabla_{X_i} Y, X_i) \\ & = -\frac{1}{2}\sum_i Yg(X_i, X_i) + \sum_i X_i g(Y, X_i) - \sum_i g(Y, \nabla_{X_i} X_i) \\ & = -g\Big(Y, \sum_i \nabla_{X_i} X_i\Big), \end{aligned}$$

$$d\omega(Y, X_1, \ldots, X_n)(p) = 0 \text{ for any } Y \in \Gamma(E^\perp) \quad \text{iff} \quad \Big(\sum_i \nabla_{X_i} X_i\Big)^\perp(p) = 0;$$

$$\begin{aligned} & d\omega(Y_1, Y_2, X_1, \ldots, \hat{X}_i, \ldots, X_n) \\ & \qquad = -\omega([Y_1, Y_2], X_1, \ldots, \hat{X}_i, \ldots, X_n) = (-1)^i g([Y_1, Y_2], X_i), \end{aligned}$$

$$d\omega(Y_1, Y_2, X_1, \ldots, \hat{X}_i, \ldots, X_n)(p) = 0 \text{ for any } i \leqslant n$$

$$\text{iff} \quad [Y_1, Y_2](p) \in E^\perp \cap T_p M.$$

COROLLARY. *Under the assumption of Theorem* 1 *and the assumption of involutivity of $E|_U$ we have the following two pairs of equivalent conditions:*

(i′) $d\omega(p) = 0$ *for any* $p \in N$,

(ii′) N *is minimal and* $E^\perp$ *is involutive at points of* N, N *being any integral manifold of* $E|_U$;

(i″) $d\omega = 0$,

(ii″) *all the integral manifolds of* $E|_U$ *are minimal and* $E^\perp|_U$ *is involutive.*

Remark 3. If the codimension of E equals 1 ($n = m-1$), then also the involutivity of $E^\perp$ is assured.

If E is orientable and minimal, and $E^\perp$ is involutive, then the global orientation form ω determines an n-dimensional de Rham cohomology class of M.

PROPOSITION. *Suppose that M is compact orientable and that there exists an orientable distribution E on M such that*

(i) E *and* $E^\perp$ *are involutive,*

(ii) *integral manifolds of* E *and* $E^\perp$ *are minimal.*

Orientability both of E and M yield orientability of $E^{\perp}$. Moreover, if ω and μ are global orientation forms of E and $E^{\perp}$, respectively, then $\omega\wedge\mu$ is a volume element of M, and so

$$[\omega]\cup[\mu]\neq 0$$

in the cohomology algebra of M.

Consequently, the cup product

$$\cup : H^n(M)\otimes H^{m-n}(M)\to H^m(M)$$

is an epimorphism.

Now we come to relations between the orientation form and the property of having totally geodesic integral manifolds.

Definition 5. A form φ over $U\subset M$ is *parallel* in a direction of E if $\nabla_X\varphi = 0$ for each $X\in E|_U$.

THEOREM 2. *If E is a distribution orientable over $U\subset M$ and ω is the orientation form, then we have the following two pairs of equivalent conditions:*

(i) *ω is parallel in a direction of E,*

(ii) *$E|_U$ is involutive and its integral manifolds are totally geodesic;*

(i′) $\nabla\omega = 0$,

(ii′) *$E|_U$ and $E^{\perp}|_U$ are involutive and their integral manifolds are totally geodesic.*

Proof. Let V, $(X_1, \ldots, X_n)$, Y, Y_1, Y_2 be as in the proof of Theorem 1. For $i\leqslant n$,

$$\begin{aligned}&(\nabla_{X_i}\omega)(X_1, \ldots, X_n)\\ &= X_i\omega(X_1, \ldots, X_n)-\omega(\nabla_{X_i}X_1, X_2, \ldots, X_n)-\ldots-\omega(X_1, \ldots, X_{n-1}, \nabla_{X_i}X_n)\\ &\qquad = -\sum_j g(\nabla_{X_i}X_j, X_j) = -\frac{1}{2}\sum_j X_i g(X_j, X_j) = 0.\end{aligned}$$

For $i, j\leqslant n$,

$$\begin{aligned}&(\nabla_{X_i}\omega)(X_1, \ldots, X_{j-1}, Y, X_{j+1}, \ldots, X_n)\\ &\qquad = -\omega(X_1, \ldots, X_{j-1}, \nabla_{X_i}Y, X_{j+1}, \ldots, X_n) = -g(\nabla_{X_i}Y, X_j)\\ &\qquad = -X_i g(Y, X_j)+g(Y, \nabla_{X_i}X_j) = g(Y, \nabla_{X_i}X_j).\end{aligned}$$

Hence

$$\nabla_{X_i}\omega = 0 \text{ for } i\leqslant n \quad \text{iff} \quad \nabla_{X_i}X_j\in\Gamma(V, E) \text{ for } i, j\leqslant n. \tag{1}$$

Since $[X_i, X_j] = \nabla_{X_i}X_j-\nabla_{X_j}X_i$, we have (i) iff (ii):

$$(\nabla_Y\omega)(X_1, \ldots, X_n) = -\sum_i g(\nabla_Y X_i, X_i) = 0.$$

For $i \leqslant n$,

$$(\nabla_{Y_1}\omega)(X_1, \ldots, X_{i-1}, Y_2, X_{i+1}, \ldots, X_n)$$
$$= -\omega(X_1, \ldots, X_{i-1}, \nabla_{Y_1}Y_2, X_{i+1}, \ldots, X_n) = -g(\nabla_{Y_1}Y_2, X_i).$$

Hence

$$(\nabla_{Y_1}\omega)(X_1, \ldots, X_{i-1}, Y_2, X_{i+1}, \ldots, X_n) = 0 \text{ for } i \leqslant n$$
$$\text{iff} \quad \nabla_{Y_1}Y_2 \in \Gamma(V, E^{\perp}).$$

Combining this with (1) we have the equivalence of conditions (i′) and (ii′).

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