COLLOQUIUM MATHEMATICUM

VOL. XLVIII 1984 FASC. 2

SEPARATION PROPERTIES AND BOOLEAN POWERS

BY

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0. Introduction. For a structure $\mathfrak{A}$ for a first order language $\mathscr{L}$ and a Boolean algebra B, $\mathfrak{A}[B]^*$ denotes the bounded Boolean power of $\mathfrak{A}$ with respect to B. We can describe $\mathfrak{A}[B]^*$ as follows: $S(B)$ is the Stone space of B. The universe of $\mathfrak{A}[B]^*$ consists of all continuous functions from $S(B)$ into $\mathfrak{A}$ (where $\mathfrak{A}$ has the discrete topology). $\mathfrak{A}[B]^*$ becomes an $\mathscr{L}$-structure by defining the operations and relations pointwise. For an n-ary relation R, this means $R(f_1, \ldots, f_n)$ if and only if $R^{\mathfrak{A}}(f_1(x), \ldots, f_n(x))$ for all $x \in S(B)$.

This version of bounded Boolean powers is equivalent to that given in [2] whose notation we have adopted. We refer the reader to [2] as a general reference on Boolean powers, and to [3] as an all-purpose source on model theory.

Definition (Burris [2]). $\mathfrak{A}$ is *B-separating* if $\mathfrak{A}[B_0]^* \cong \mathfrak{A}[B_1]^*$ implies $B_0 \cong B_1$.

Definition. $\mathfrak{A}$ is *weakly separating* if $\mathfrak{A}[B_0]^* \equiv \mathfrak{A}[B_1]^*$ implies $B_0 \equiv B_1$.

Briefly, the following is known: A Boolean algebra B is weakly separating iff the Tarski–Ershov invariant of B is $\langle m, n\rangle$, where m is even and $0 < n \leqslant \infty$ (see [7]); a finite distributive lattice is weakly separating [6]; bounded chains are both B-separating and weakly separating (see [1] and [9]). These results rely on the isomorphism between bounded Boolean powers of bounded distributive lattices and free products described in [8] and [9].

In addition, Jónsson [5] has shown that a centerless indecomposable group is B-separating. Recent results of John Lawrence (a letter from S. Burris) show that no abelian group is B-separating or weakly separating.

In Section 1 we show that every chain with more than one element is both B-separating and weakly separating thereby generalizing the results of Balbes and Dwinger [1] and Speed [9]. In Section 2 we describe an $\mathfrak{A}$ which is B-separating but not weakly separating. This provides a negative answer to a question of Burris in [2].

1. Chains. We identify a Boolean algebra B with the algebra of clopen subsets of its Stone space $S(B)$. For $a \in B$, $B|_a$ denotes the restriction of B to a; i.e., $B|_a$ consists of all elements of B which are less than or equal to a with the induced operations. It is well known and easy to prove that $B|_a$ is a direct factor of B. In fact, if a' is the complement of a, then $B|_a \times B|_{a'} = B$.

For L a bounded distributive lattice, $C(L)$ denotes the Boolean algebra of complemented elements of L. If L is any distributive lattice and $x, y \in L$, $x \leqslant y$, $x \neq y$, then $L[x, y]$ is the bounded distributive lattice of all $z \in L$ with $x \leqslant z \leqslant y$ and with the induced operations.

It is easy to see that if $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ is a (bounded, distributive, Boolean) lattice, then so is $\mathfrak{A}[B]^*$.

LEMMA 1. *Let $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ be a chain, B a Boolean algebra, $f, g \in L = \mathfrak{A}[B]^*$, $f \neq g$ and $f \leqslant g$. Let*

$$D = \{x \in S(B) \mid f(x) \neq g(x)\}.$$

Then $C(L[f, g]) \cong B|_D$.

Proof. The following preliminary observation is useful: if $h \in C(L[f, g])$, then for every $x \in S(B)$ either $h(x) = f(x)$ or $h(x) = g(x)$. This is true because if $f(x) < h(x) < g(x)$ and h' is any element of $L[f, g]$, then one cannot simultaneously have $h(x) \wedge h'(x) = f(x)$ and $h(x) \vee h'(x) = g(x)$ since $\mathfrak{A}$ is a chain. So h would not be complemented in $L[f, g]$, contrary to our assumption.

Now, let $h \in C(L[f, g])$. Let $D_h = \{x \in S(B) \mid h(x) \neq f(x)\}$. It follows immediately from the preceding remarks that the map $h \to D_h$ of $C(L[f, g])$ into $B|_D$ is 1-1. Given any element U of $B|_D$, i.e., a clopen subset of D, define h_U by

$$h_U(x) = \begin{cases} g(x), & x \in U, \\ f(x), & x \notin U. \end{cases}$$

Then $h_U \in C(L[f, g])$ and $D_{h_U} = U$. Thus the map $h \to D_h$ is onto. It is easy to see it is also a Boolean algebra isomorphism.

THEOREM 1. *Every chain with more than one element is B-separating.*

Proof. Let $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ be a chain with at least two elements. Let B_0, B_1 be Boolean algebras and let φ be an isomorphism of $L_0 = \mathfrak{A}[B_0]^*$ onto $L_1 = \mathfrak{A}[B_1]^*$. Choose $a < b$ in $\mathfrak{A}$ and define $f(x) = a$ and $g(x) = b$ for all $x \in S(B_0)$. Now, find f_1 and g_1 in $\mathfrak{A}[B_1]^*$ such that $f_1 \leqslant \varphi(f) \leqslant \varphi(g) \leqslant g_1$ and $f_1(x) \neq g_1(x)$ for all $x \in S(B_1)$.

By Lemma 1, $C(L_1[f_1, g_1]) \cong B_1|_{S(B_1)} = B_1$. Also we have

$$\varphi^{-1}(f_1) \leqslant f \leqslant g \leqslant \varphi^{-1}(g_1).$$

Thus $\varphi^{-1}(f_1)(x) \neq \varphi^{-1}(g_1)(x)$ for all $x \in S(B_0)$. Consequently, by Lemma 1,

$$C(L_0[\varphi^{-1}(f_1), \varphi^{-1}(g_1)]) \cong B_0.$$

Clearly, φ maps $C(L_0[\varphi^{-1}(f_1), \varphi^{-1}(g_1)])$ isomorphically onto $C(L_1[f_1, g_1])$, so $B_0 \cong B_1$.

The proof that nontrivial chains are weakly separating is somewhat different. It is convenient to use an interesting but not very well known result of Galvin about factor sentences. A sentence ψ (in the appropriate first order language) is a *factor sentence* if for all structures $\mathfrak{A}_0$, $\mathfrak{A}_1$ the relation $\mathfrak{A}_0 \times \mathfrak{A}_1 \models \psi$ implies $\mathfrak{A}_0 \models \psi$ and $\mathfrak{A}_1 \models \psi$. Galvin [4] proved that if for every factor sentence ψ we have $\mathfrak{A}_0 \models \psi$ if and only if $\mathfrak{A}_1 \models \psi$, then $\mathfrak{A}_0 \equiv \mathfrak{A}_1$.

THEOREM 2. *Every chain with more than one element is weakly separating.*

Proof. Let $\mathfrak{A} = \langle A, \wedge, \vee \rangle$ be a nontrivial chain and let B_0, B_1 be Boolean algebras with $\mathfrak{A}[B_0]^* \equiv \mathfrak{A}[B_1]^*$. Let ψ be a factor sentence in the language of Boolean algebras and suppose $B_0 \models \psi$. Suppose, in order to reach a contradiction, that $B_1 \models \neg\psi$. Choose $f, g \in \mathfrak{A}[B_1]^*$ such that $f \leqslant g$ and $f(x) \neq g(x)$ for all $x \in S(B_1)$. Then for $L_1 = \mathfrak{A}[B_1]^*$ we have $C(L_1[f, g]) \cong B_1$ and, consequently,

$$\mathfrak{A}[B_1]^* \models \text{“}\exists v \exists w [v \leqslant w \wedge v \neq w \wedge C(L_1[v, w]) \models \neg\psi]\text{”}.$$

The formula in quotation marks is expressible in the first order theory of lattices; since $\mathfrak{A}[B_0]^* \equiv \mathfrak{A}[B_1]^*$, there are $f_0, g_0 \in L_0 = \mathfrak{A}[B_0]^*$ such that $f_0 \leqslant g_0$, $f_0 \neq g_0$, and $C(L_0[f_0, g_0]) \models \neg\psi$. But, by Lemma 1, $C(L_0[f_0, g_0])$ is a factor of B_0. Since ψ is a factor sentence and $B_0 \models \psi$, we have $C(L_0[f_0, g_0]) \models \psi$. This contradiction shows that every factor sentence true in B_0 is true in B_1. Reversing the roles of B_0 and B_1, we infer that every factor sentence true in B_1 is true in B_0. Consequently, by Galvin's theorem, $B_0 \equiv B_1$.

Remarks. The proof of the preceding theorem could be carried out by examining the Tarski–Ershov invariants of B_0 and B_1, but the use of factor sentences seems to simplify the argument. Theorems 1 and 2 hold if the chain $\mathfrak{A}$ has an additional structure. In particular, if $\mathfrak{A}$ is an ordered group, then $\mathfrak{A}[B]^*$ is a lattice ordered group. And if $\mathfrak{A}[B_0]^* \equiv \mathfrak{A}[B_1]^*$ (in the language of lattices), then $B_0 \equiv B_1$. This answers a question of A. M. W. Glass.

2. An example. In [2] Burris asked the following question: If $\mathfrak{A}$ is B-separating and $\mathfrak{A}[B_0]^* \equiv \mathfrak{A}[B_1]^*$, does it follow that $B_0 \equiv B_1$? (I.e., is $\mathfrak{A}$ weakly separating?)

Let $\mathscr{L}$ be the language $\{\wedge, \vee, ', 0, 1\}$ for Boolean algebras with countably many unary predicates $\{U_n \mid n \in \omega\}$ added. We show there is a countable $\mathscr{L}$-structure $\mathfrak{A} = \langle B, U_n, 0, 1\rangle_{n\in\omega}$ such that

(1) B and all the U_n's are atomless Boolean algebras. The U_n's form a descending chain of subalgebras of B and

$$\bigcap_{n\in\omega} U_n = \{0, 1\}.$$

(2) $\mathfrak{A} \times \mathfrak{A} \equiv \mathfrak{A}$.

Remarks. The properties in (1) can be expressed by a single sentence of $\mathscr{L}_{\omega_1\omega}$.

$\mathfrak{A}$ provides a negative answer to Burris's question.

THEOREM 3. *If $\mathfrak{A}$ satisfies (1) and (2), then $\mathfrak{A}$ is B-separating but not weakly separating.*

Proof. By (1), $\{f \in \mathfrak{A}[B_0]^* \mid \mathfrak{A}[B_0]^* \models U_n[f] \text{ for all } n\}$ equals

$$\{f \in \mathfrak{A}[B_0]^* \mid \text{range } f \subseteq \{0, 1\}\},$$

which is isomorphic to B_0. Thus $\mathfrak{A}[B_0]^* \cong \mathfrak{A}[B_1]^*$ implies $B_0 \cong B_1$. And $\mathfrak{A}$ is B-separating. From (2) it follows that $\mathfrak{A}$ fails to weakly separate even finite Boolean algebras:

$$\mathfrak{A}[2^{2n}]^* \cong \mathfrak{A} \times \ldots \times \mathfrak{A} \,(2n \text{ times}) \equiv \mathfrak{A} \times \ldots \times \mathfrak{A} \,(n \text{ times}) \cong \mathfrak{A}[2^n]^*$$
$$\text{while } 2^{2n} \not\equiv 2^n.$$

The existence of $\mathfrak{A}$. For $1 \leqslant m < \omega$ let $\mathfrak{A}_m$ be the $\mathscr{L}$-structure $\langle B_m, U_n^m, 0, 1\rangle_{n\in\omega}$, where B_m is an atomless Boolean algebra of cardinality ω_m which is ω_m-saturated. In addition, for $n < m$, U_n^m is an ω_{m-n-1}-saturated atomless Boolean subalgebra of B_m of cardinality ω_{m-n-1}; for $n \geqslant m$, $U_n^m = \{0, 1\}$. For all n, we have $U_n^m \supseteq U_{n+1}^m$. To prove the existence of such Boolean algebras with prescribed cardinality and degree of saturation we use the GCH.

Let $\mathscr{L}_m$ be $\mathscr{L} \restriction \{U_0, \ldots, U_{m-1}\}$. We have

$$\mathfrak{A}_m \restriction \mathscr{L}_m \times \mathfrak{A}_m \restriction \mathscr{L}_m = \langle B_m \times B_m, U_0^m \times U_0^m, \ldots, U_{m-1}^m \times U_{m-1}^m\rangle.$$

Each of these direct products is an atomless Boolean algebra of the same cardinality and degree of saturation as $B_m, U_0^m, \ldots, U_{m-1}^m$, respectively. Since any two countable atomless Boolean algebras are isomorphic, we have $U_{m-1}^m \times U_{m-1}^m \cong U_{m-1}^m$. Using the ω_1-saturation of $U_{m-2}^m \times U_{m-2}^m$ and U_{m-2}^m, we extend the above isomorphism to get $U_{m-2}^m \times U_{m-2}^m \cong U_{m-2}^m$. Saturation allows us to continue to extend isomorphisms until we have

(3) $\mathfrak{A}_m \restriction \mathscr{L}_m \times \mathfrak{A}_m \restriction \mathscr{L}_m \cong \mathfrak{A}_m \restriction \mathscr{L}_m$.

LEMMA 2. *Let $\varphi(v) \in \mathscr{L}_m$ and $k > m$. If*

$$\mathfrak{A}_k \models \exists v\,[\varphi(v) \wedge \neg v = 0 \wedge \neg v = 1],$$

then $\mathfrak{A}_k \models \exists v\,[\varphi(v) \wedge \neg U_m(v)]$.

Proof. Let $b \in \mathfrak{A}_k$, $b \neq 0, 1$, with $\mathfrak{A}_k \models \varphi[b]$. Assume $b \in U_m^k$. Let $c \in U_{m-1}^k$, $c \notin U_m^k$. Such a c exists by cardinality considerations. Now, b, c are both different from 0, 1 and are elements of the saturated atomless Boolean algebra U_{m-1}^k. The mapping which sends b to c can be extended to an automorphism of U_{m-1}^k. And this again can be extended by the repeated use of saturation to an automorphism of $\mathfrak{A}_k \restriction \mathscr{L}_m$. Thus

$$\mathfrak{A}_k \models \varphi[c] \wedge \neg U_m[c].$$

Now, let $\mathscr{U}$ be a nonprincipal ultrafilter on $\omega - \{0\}$. Let $\mathfrak{A}'$ be the ultraproduct $\prod_{m\geq 1} \mathfrak{A}_m/\mathscr{U}$. From (3) it follows easily that $\mathfrak{A}' \times \mathfrak{A}' \cong \mathfrak{A}'$. The ultraproduct $\mathfrak{A}'$ is far from countable but our goals of (1) and (2) are almost achieved. Of course, we have no knowledge of $\bigcap_{n\in\omega} U'_n$.

Consider the theory of $\mathfrak{A}'$; i.e.,

$$T = \{\varphi \in \mathscr{L} \mid \varphi \text{ sentence and } \mathfrak{A}' \models \varphi\}.$$

What we want is a countable model $\mathfrak{A}$ of T which omits the set of formulas

$$\Sigma(v) = \{\neg v = 0, \ \neg v = 1, \ U_n(v) \mid n \in \omega\}.$$

For such an $\mathfrak{A}$ we would have $\mathfrak{A} \equiv \mathfrak{A}'$, and since $\equiv$ is preserved by direct products ([3], p. 345), $\mathfrak{A} \times \mathfrak{A} \equiv \mathfrak{A}$. Moreover, since $\mathfrak{A}$ omits $\Sigma(v)$, $\bigcap_{n\in\omega} U_n = \{0, 1\}$. Then $\mathfrak{A}$ would satisfy (1) and (2).

We obtain $\mathfrak{A}$ by applying the omitting types theorem ([3], p. 79). We proceed to show T locally omits $\Sigma(v)$. Let $\varphi(v)$ be a formula of $\mathscr{L}$ consistent with T. Then $\mathfrak{A}' \models \exists v(\varphi)$. If 1 or 0 satisfies φ in $\mathfrak{A}'$, we are done. Assume

$$\mathfrak{A}' \models \exists v[\varphi(v) \wedge \neg v = 0 \wedge \neg v = 1].$$

Thus

$$X = \{k \mid \mathfrak{A}_k \models \exists v[\varphi(v) \wedge \neg v = 0 \wedge \neg v = 1] \in \mathscr{U}\}.$$

By Lemma 2 there is an m such that, for all but a finite number of k in X, $k \in X$ implies

$$\mathfrak{A}_k \models \exists v[\varphi(v) \wedge \neg U_m(v)].$$

Since $\mathscr{U}$ is nonprincipal,

$$\mathfrak{A}' \models \exists v[\varphi(v) \wedge \neg U_m(v)].$$

Hence $\varphi(v) \wedge \neg U_m(v)$ is consistent with T and T locally omits $\Sigma(v)$. The existence of $\mathfrak{A}$ now follows from the omitting types theorem.

Remarks. At one point we used the GCH. All required properties of $\mathfrak{A}$ are absolute for standard models of set theory. $\mathfrak{A}$ certainly exists in the constructible universe, and thus it exists by absoluteness. The dependence on the GCH can thereby be eliminated.

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Reçu par la Rédaction le 21. 3. 1980