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THE RATE OF CONVERGENCE OF ITERATES OF THE FROBENIUS-PERRON OPERATOR FOR PIECEWISE MONOTONIC TRANSFORMATIONS

BY

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1. Introduction. Lasota [5] and Jabłoński [2] have shown that the sequence $P_\tau^n f$ of iterates of the Frobenius-Perron operator $P_\tau: L^1 \to L^1$ given by an expanding transformation $\tau: M \to M$ of the differentiable manifold into itself is uniformly convergent for a certain class of functions $f: M \to R$. A similar theorem has also been stated by Krzyżewski [3]. But the above authors have assumed in their theorems that the transformation τ is continuous. The purpose of this note is to give an analogous theorem in the case where the transformation $\tau: [0, 1] \to [0, 1]$ is piecewise C^2 with some point of discontinuity.

The paper is divided into two parts. In Section 2 we give the convergence theorem for the Frobenius-Perron operator $P_\tau: L^1(m) \to L^1(m)$, where m is a Lebesgue measure. In Section 3 we prove a theorem concerning the rate of convergence of the iterates of the Frobenius-Perron operator $P_\tau: L^1(\mu) \to L^1(\mu)$, where μ is an absolutely continuous measure invariant under τ.

2. Convergence theorem for τ · piecewise C^2. Let $([0, 1], \Sigma, \nu)$ be a probability space with measure ν and let $L^1([0, 1], \Sigma, \nu)$ be the space of all integrable functions defined on $[0, 1]$. For a nonsingular transformation $\tau: [0, 1] \to [0, 1]$ $(\nu(\tau^{-1}(A)) = 0$ whenever $\nu(A) = 0)$ we define the Frobenius-Perron operator

$$P_\tau: L^1([0, 1], \Sigma, \nu) \to L^1([0, 1], \Sigma, \nu)$$

by the formula

$$\int_A P_\tau f d\nu = \int_{\tau^{-1}(A)} f d\nu,$$

which is valid for each measurable set $A \subset [0, 1]$. It is well known that the operator P_τ is linear and continuous and satisfies the following conditions:

(a) P_τ is positive: $f \geqslant 0 \Rightarrow P_\tau f \geqslant 0$;

(b) P_τ preserves integrals:

$$\int_0^1 P_\tau f dv = \int_0^1 f dv, \quad f \in L^1(v);$$

(c) $P_{\tau^n} = P_\tau^n$ (τ^n denotes the n-th iterate of τ);

(d) $P_\tau f = f$ if and only if the measure $d\mu = f dv$ is invariant under τ, that is $\mu(\tau^{-1}(A)) = \mu(A)$ for each measurable A.

In the sequel we denote, for convenience, by $\bar{P}_\tau$ and P_τ the Frobenius-Perron operators defined on $L^1([0, 1], \Sigma, \mu)$ and $L^1([0, 1], \Sigma, m)$, respectively.

A transformation τ: $[0, 1] \to R$ will be called *piecewise* C^2 if there exists a partition $0 = a_0 < a_1 < \ldots < a_p = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, p$) the restriction τ_i of τ to the open interval (a_{i-1}, a_i) is a C^2-function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2-function. τ need not be continuous at the points a_i.

Lasota and Yorke [6] have proved that for a piecewise C^2-transformation τ with $\inf|\tau'| > 1$ there exists an absolutely continuous measure μ invariant under τ and the density of μ is of bounded variation.

Denote by

$$\bigvee_a^b f = \bigvee_{[a,b]} f$$

the variation of f over the interval $[a, b]$.

We shall show the following convergence theorem:

THEOREM 1. *Let* τ: $[0, 1] \to [0, 1]$ *be a piecewise* C^2*-function such that*

$$s = \sup_{i,x} \left|\frac{\varphi_i''(x)}{\varphi_i'(x)}\right| + \sup_{i,x} |\varphi_i'(x)| \left(1 + \sum_{i=1}^{p} \delta_i\right) < 1, \tag{1}$$

where $\delta_i = 2 - \operatorname{card}\{\tau(a_i+), \tau(a_i-)\} \cap \{0, 1\}$ *and* $\varphi_i = \tau_i^{-1}$ $(i = 1, 2, \ldots, p)$. *Then there exists exactly one probabilistic, absolutely continuous measure* μ *invariant under* τ *and for any* $f \geqslant 0$ *with bounded variation we have*

$$|(P_\tau^n f)(x) - \|f\| f_0(x)| \leqslant s^n \left(\bigvee_0^1 f + \|f\| \bigvee_0^1 f_0\right), \tag{2}$$

where f_0 *is the density of the measure* μ.

Proof. A simple computation shows that the Frobenius-Perron operator P corresponding to τ may be written in the form

$$(P_\tau f)(x) = \sum_{i=1}^{p} f(\varphi_i(x)) |\varphi_i'(x)| \chi_i(x), \tag{3}$$

where χ_i is the characteristic function of the interval $J_i = \tau_i([a_{i-1}, a_i])$. By its very definition the operator P_τ is a mapping from $L^1([0, 1], \Sigma, m)$ into $L^1([0, 1], \Sigma, m)$, but the last formula enables us to consider P_τ as a map from the space of functions defined on $[0, 1]$ into itself.

Let f be a function with bounded variation such that

$$\int_0^1 f dm = 0.$$

We have

$$\bigvee_0^1 P_\tau f \leqslant \sum_{i=1}^{p} \bigvee_{J_i} (f \circ \varphi_i)|\varphi_i'| + \sup_{i,x} |\varphi_i'| \sum_{i=1}^{p} \delta_i |f(a_i)|. \tag{4}$$

In order to evaluate the first sum we write

$$\bigvee_{J_i} (f \circ \varphi_i)|\varphi_i'| = \int_{J_i} |d((f \circ \varphi_i)|\varphi_i'|)| \leqslant \int_{J_i} |f \circ \varphi_i| \, |\varphi_i''| \, dm + \int_{J_i} |\varphi_i'| \, |d(f \circ \varphi_i)|$$

$$\leqslant \sup_{i,x} \left| \frac{\varphi_i''(x)}{\varphi_i'(x)} \right| \int_{J_i} |f \circ \varphi_i| \, |\varphi_i'| \, dm + \sup_{i,x} |\varphi_i'(x)| \int_{J_i} |d(f \circ \varphi_i)|.$$

Changing the variables we obtain

$$\bigvee_{J_i} (f \circ \varphi_i)|\varphi_i'| \leqslant \sup_{i,x} \left| \frac{\varphi_i''(x)}{\varphi_i'(x)} \right| \int_{a_{i-1}}^{a_i} |f| \, dm + \sup_{i,x} |\varphi_i'(x)| \bigvee_{a_{i-1}}^{a_i} f. \tag{5}$$

Since $\int_0^1 f dm = 0$, we have the obvious inequality

$$|f(x)| \leqslant \bigvee_0^1 f. \tag{6}$$

Applying (6) and (5) to (4) we obtain

$$\bigvee_0^1 P_\tau f \leqslant s \bigvee_0^1 f$$

and, consequently, by induction we have

$$\bigvee_0^1 P_\tau^n f \leqslant s^n \bigvee_0^1 f. \tag{7}$$

Now, we shall show that the absolutely continuous measure μ invariant under τ is unique. To the contrary, assume that μ_1 and μ_2 are two different probabilistic measures invariant under τ with densities f_1 and f_2, respectively. Since

$$\int_0^1 (f - \|f\| f_1) dm = 0 \quad \text{and} \quad \int_0^1 (f - \|f\| f_2) dm = 0,$$

by (7) for f of bounded variation we have

$$\bigvee_0^1 P_\tau^n (f-\|f\| f_1) \leqslant s^n (\bigvee_0^1 f + \|f\| \bigvee_0^1 f_1)$$

and

$$\bigvee_0^1 P_\tau^n (f-\|f\| f_2) \leqslant s^n (\bigvee_0^1 f + \|f\| \bigvee_0^1 f_2),$$

which is impossible because each convergent sequence has only one limit. Since

$$\int_0^1 (f-\|f\| f_0)\, dm = 0,$$

inequality (2) is a simple consequence of (6), (7), and (d). This completes the proof of the theorem.

3. Convergence theorem for τ piecewise convex.

THEOREM 2. *If τ is a piecewise C^2-transformation of the unit interval into itself such that*

(i) $\tau_i = \tau|_{[a_{i-1}, a_i]}$ $(i = 1, 2, \ldots, p)$ *are convex,*

(ii) $\tau_i(a_{i-1}) = 0$ $(i = 1, 2, \ldots, p)$,

(iii) $\tau([a_0, a_1]) = [0, 1]$,

(iv) τ *satisfies* (1),

then there exists a unique τ-invariant absolutely continuous measure μ with density $f_0 = d\mu/dm$ satisfying the inequality

(8) $$0 < 1/c \leqslant f_0 \leqslant c \quad \textit{for some } c.$$

Proof. The uniqueness of μ is a consequence of Theorem 1. Now, we show (8). By induction, from (i), (ii), and (3) it follows that $P_\tau^n f$ are decreasing functions whenever f is decreasing. Consequently, by Theorem 1, there exists a sequence $P_\tau^n f$ of decreasing functions convergent to f_0. Therefore, f_0 is decreasing, and so $\operatorname{supp} f_0 = [0, a]$ for some $a \leqslant 1$. Now, applying the inclusion (see [10])

$$\tau(\operatorname{supp} f_0) \subset \operatorname{supp} f_0$$

we can easily seen that

(9) $$\operatorname{supp} f_0 = [0, 1].$$

Since f_0 is decreasing, by (d), (3), and (9) we have

$$f_0(x) \geqslant f_0(a_1)\varphi_1'(a_1) > 0 \quad \text{for } x \in [0, 1].$$

This completes the proof because f_0 is of bounded variation.

THEOREM 3. *If τ satisfies the assumptions of Theorem 2, then for any $f \geqslant 0$ of bounded variation we have*

$$|(\bar{P}_\tau^n f)(x) - \|f\|_{L^1(\mu)}| \leqslant s^n \big(M_1 \|f\|_{L^1(\mu)} + M_2 \bigvee_0^1 f\big),$$

where $M_1 = 2c \bigvee_0^1 f_0$, $M_2 = (\bigvee_0^1 f_0 + c)c$, $\bar{P}_\tau$ is the Frobenius-Perron operator of the space $L^1(\mu)$ into itself, and μ is a measure invariant under τ.

Proof. By the equalities

$$\int_A \bar{P}_\tau f d\mu = \int_A (\bar{P}_\tau f) f_0 dm$$

and

$$\int_A \bar{P}_\tau f d\mu = \int_{\tau^{-1}(A)} f d\mu = \int_{\tau^{-1}(A)} f f_0 dm = \int_A P_\tau (f f_0) dm$$

we obtain

$$(\bar{P}_\tau f) f_0 = P_\tau (f f_0)$$

and, consequently,

$$\bar{P}_\tau f = \frac{P_\tau (f f_0)}{f_0}. \tag{10}$$

Let $f \geqslant 0$ be a function of bounded variation. By Theorems 1 and 2 and (10) we have

$$|\bar{P}_\tau^n f - \|f\|_{L^1(\mu)}| = \left|\frac{P_\tau^n (f f_0) - \|f f_0\|_{L^1(m)} f_0}{f_0}\right| \leqslant s^n c \big(\bigvee_0^1 f f_0 + \|f f_0\|_{L^1(m)} \bigvee_0^1 f_0\big).$$

Since

$$\bigvee_0^1 f f_0 \leqslant (\sup f) \bigvee_0^1 f_0 + (\sup f_0) \bigvee_0^1 f \leqslant (\|f\|_{L^1(\mu)} + \bigvee_0^1 f) \bigvee_0^1 f_0 + c \bigvee_0^1 f$$

and $\|f f_0\|_{L^1(m)} = \|f\|_{L^1(\mu)}$, from the last inequality we obtain

$$|\bar{P}_\tau^n f - \|f\|_{L^1(\mu)}| \leqslant s^n c \big[\bigvee_0^1 f \big(\bigvee_0^1 f_0 + c\big) + 2 \|f\|_{L^1(\mu)} \bigvee_0^1 f_0\big]$$
$$\leqslant s^n \big[M_1 \|f\|_{L^1(\mu)} + M_2 \bigvee_0^1 f\big].$$

This completes the proof.

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