COLLOQUIUM MATHEMATICUM

VOL. XLV 1981 FASC. 2

ON SETS OF COMPLETELY UNIFORM CONVERGENCE

BY

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1. In this paper*, $\boldsymbol{T}$ denotes the circle group, $\boldsymbol{Z}$ the group of the integers, and $M(\boldsymbol{T})$ the space of all complex regular Borel measures on $\boldsymbol{T}$. For $E \subseteq \boldsymbol{Z}$, $C_E(\boldsymbol{T})$ denotes the space of all continuous E-spectral functions on $\boldsymbol{T}$.

A subset E of $\boldsymbol{Z}$ is called a *UC-set* if every function in $C_E(\boldsymbol{T})$ has uniformly convergent Fourier series. Clearly, every Sidon set is a UC-set, while the converse is not true as Figà-Talamanca first showed [2]. For a detailed study and examples of UC-sets we refer to [7] and [9].

We define the UC-constant of a set $E \subseteq \boldsymbol{Z}$ to be

$$\varkappa(E) = \operatorname{Sup}\{\|S_N f\|_\infty/\|f\|_\infty \colon f \in C_E(\boldsymbol{T}), f \neq 0, N \in \boldsymbol{Z}^+\},$$

where

$$S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)\exp(inx).$$

Clearly, E is a UC-set if and only if $\varkappa(E)$ is finite.

Definition 1. A set $E \subset \boldsymbol{Z}$ is a *set of completely uniform convergence* (a *CUC-set*) if $\operatorname{Sup}\{\varkappa(\{E+p\})\colon p \in \boldsymbol{Z}\} < \infty$, where $\{E+p\} = \{q+p \colon q \in E\}$.

This definition was introduced by Ricci in [8]. CUC-sets turn out to be related to subsets of $\boldsymbol{T}$ which are not Helson sets and do not carry unbounded pseudomeasures ([8], Chapter 4).

It is easy to see that a UC-set contained in the set of positive integers is a CUC-set (see [8], Proposition 4.4, and [9]). A question implicitly raised by F. Ricci is whether there exist (non-Sidon) CUC-sets which are essentially different from this example. In this paper we give an affirmative answer by constructing a symmetric non-Sidon CUC-set. Moreover, we show that if there exists a UC-set which is not a CUC-set, then the union problem for UC-sets has a negative answer. As a corollary to the proof,

* This work was partially supported by the C.N.R.

there exist non-UC-sets which do not contain arbitrarily long arithmetic progressions.

2. Arguing as in [7], Theorem 1, it is easy to prove the following proposition (see [8], Proposition 4.4, and [9], p. 283).

PROPOSITION 1. *A UC-set E is a CUC-set if and only if there exists a measure $\mu \in M(T)$ such that $\hat{\mu}|(E\cap Z^+) = 1$ and $\hat{\mu}|(E\cap Z^-) = 0$.*

The proposition relates CUC-sets with the notion of harmonic separation introduced in [3] and further discussed in [5] and [4].

Definition 2. Two distinct subsets A and B of Z are *harmonically separated* if there exists a measure $\mu \in M(T)$ such that $\hat{\mu}|A = 1$ and $\hat{\mu}|B = 0$.

We remark that, by Drury's theorem [1], any two disjoint Sidon sets are harmonically separated, while this is not true for UC-sets ([9], Lemma 4).

3. We have the following result:

THEOREM. *There exists a symmetric CUC-set which is not a Sidon set.*

Proof. We begin with the observation that for any interval $(u, v) \subset [0, 2\pi)$ there exists a (Hadamard) set $\{n_k\}_{k=1}^{\infty} \subset Z^+$ such that

$$\operatorname{Inf}\{n_{k+1}/n_k\colon k\in Z^+\} \geqslant 2, \tag{1}$$

$$\overline{\{\exp(in_k)\}} \subset (u, v) \tag{2}$$

(since the closure of $\{\exp(in)\}$ coincides with the unit circle).

We choose two intervals $A, B \subset [0, 2\pi)$ such that

$$A+B \subset \left[\frac{\pi}{4}, \frac{3}{4}\pi\right],$$

where $A+B = \{p+q\colon p\in A,\ q\in B\}$.

Let Q and R be Hadamard sets which satisfy (1) and (2) with $(u, v) = A$ and $(u, v) = B$, respectively.

Let $E = Q+R$. The set E is not Sidon (see, e.g., [6], Theorem 1.4). However, by [9], Theorem 7, E is a UC-set and, by [9], Theorem 2, $E\cup(-E)$ is a UC-set.

Now we show that E and $-E$ are harmonically separated. Indeed, let $\varphi(n) = \exp(in)$; then

$$\varphi|E \subset \left[\frac{\pi}{4}, \frac{3}{4}\pi\right] \quad \text{and} \quad \varphi|-E \subset \left[\frac{5}{4}\pi, \frac{7}{4}\pi\right].$$

Let A and $-A$ be the closures of E and $-E$, respectively, in the Bohr compactification bZ of Z. We can extend φ to a continuous function $\tilde{\varphi}$ on bZ, and by continuity we have

$$\tilde{\varphi}|E \subset \left[\frac{\pi}{4}, \frac{3}{4}\pi\right] \quad \text{and} \quad \tilde{\varphi}|-E \subset \left[\frac{5}{4}\pi, \frac{7}{4}\pi\right].$$

Hence $A \cap (-A) = \emptyset$.

Let $V \subset b\boldsymbol{Z}$ be a symmetric neighborhood of 0 such that

$$\overline{(A+V)} \cap \overline{(-A+V)} = \emptyset.$$

Let $U = \overline{A+V}$ and let ψ be a continuous function on $b\boldsymbol{Z}$ such that $\psi | U = 1$ and $\psi | -U = 0$. Let χ_V and $m(V)$ denote the characteristic function and the measure of V, respectively.

Then

$$\xi = \frac{1}{m(V)} \psi * \chi_V \in A(b\boldsymbol{Z}).$$

Hence $\xi | \boldsymbol{Z}$ is the Fourier-Stieltjes transform of a discrete measure $\mu \in M(\boldsymbol{T})$.

We have

$$\xi(t) = \frac{1}{m(V)} \int_V \psi(t-z)\,dz.$$

Hence $\hat{\mu} | E = 1$ and $\hat{\mu} | -E = 0$. Then, by Proposition 1, $E \cup (-E)$ is a CUC-set.

We remark that by the same technique it is possible to construct "large" symmetric sets (of positive density) whose positive and negative parts are harmonically separated.

4. We consider the following open questions (1):

(a) Does there exist a UC-set which is not a CUC-set?

(b) Is the union of two UC-sets again a UC-set?

We can prove the following result:

PROPOSITION 2. *A positive answer to* (a) *implies a negative answer to* (b).

Proof. Let E be a UC-set which is not a CUC-set.

We write

$$E_n^+ = E \cap (0, n], \quad E_n^- = E \cap [-n, 0),$$

$$F_n^+ = \{E_n^+ + 2^{2n}\}, \quad F_n^- = \{E_n^- + 2^{2n}\},$$

$$H^+ = \bigcup_{n=1}^{\infty} F_n^+, \quad H^- = \bigcup_{n=1}^{\infty} F_n^-.$$

By [9], Theorem 3, H^+ and H^- are UC-sets. $H^+ \cup H^-$ is not a UC-set. Otherwise, by [7], Theorem 1, there exists a sequence $\{\mu_n\}$ in $M(\boldsymbol{T})$ such that

$$\hat{\mu}_n(j) = \begin{cases} 1 & \text{if } j \in (H^+ \cup H^-) \cap (0, n], \\ 0 & \text{if } j \in (H^+ \cup H^-) \setminus (0, n], \end{cases} \quad \operatorname{Sup}\{\|\mu_n\|_1 : n \in \boldsymbol{Z}^+\} < \infty.$$

(1) (a) has been solved in the positive (hence (b) in the negative) by J. J. F. Fournier (Note of the Editors).

If

$$\nu_k = \mu_{2^{2k}} \cdot \exp(-i \cdot 2^{2k} x),$$

then

$$\hat{\nu}_k(j) = \begin{cases} 1 & \text{if } j \in E \cap [-k, 0), \\ 0 & \text{if } j \in E \cap (0, k]. \end{cases}$$

By Alaoglu's theorem we have a measure $\nu \in M(\boldsymbol{T})$ such that $\hat{\nu} | (E \cap \boldsymbol{Z}^-) = 1$ and $\hat{\nu} | (E \cap \boldsymbol{Z}^+) = 0$. Then E is a CUC-set, which contradicts our assumption.

There are many non-trivial examples of subsets of $\boldsymbol{Z}$ whose positive and negative parts are not harmonically separated (see, e.g., [4]): for instance the set $\{\pm n^2\}$. This fact and the technique of the above proof allow us to construct examples of non-UC-sets which do not contain arbitrarily long arithmetic progressions.

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Reçu par la Rédaction le 20. 10. 1979