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A COUNTER-EXAMPLE ON UNICOHERENT PEANO SPACES

BY

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1. Introduction. In this article we give an example of a sequence of disjoint closed sets $A_1, A_2, \ldots$ in a unicoherent Peano space X such that $X - A_n$ is connected for each n, and yet $X - \bigcup_{n=1}^{\infty} A_n$ is not connected. This example is described in Section 3, and in Section 4 it is proved that it has the stated properties. In Section 5 we raise a question which arises from this example and the paper of van Est [4]. In Section 2 we explain the significance of the example.

2. A *Peano space* is a locally compact, connected and locally connected metric space. A *Peano continuum* is a compact Peano space. A connected space is said to be *unicoherent* if however it is expressed as the union of two connected closed subsets A and B, $A \cap B$ is always connected. We then have the following well-known theorem:

If X is a unicoherent Peano continuum and $A_1, A_2, \ldots$ is a sequence of disjoint closed subsets of X no one of which separates X, then $\bigcup_{n=1}^{\infty} A_n$ does not separate X.

This theorem has also been proved for certain non-compact unicoherent Peano spaces. In 1923 Miss Mullikin proved it in [3] for the case in which X is the plane (this proof was considerably simplified in 1924 by Mazurkiewicz in [2]), and in 1952 van Est proved it in [4] for the case in which X is a Euclidean space of any (finite) dimension. Our example shows that the theorem does not hold when X is an arbitrary Peano space.

The proof of the theorem that has been quoted was shown to me by Dr. G. T. Whyburn, and runs briefly as follows. If on the contrary $\bigcup_{n=1}^{\infty} A_n$ separates X, then it follows from the unicoherence of X that some subset F of $\bigcup_{n=1}^{\infty} A_n$ which is closed and connected in X also separates X. But now F is a continuum which can be decomposed into the sequence of disjoint closed sets $A_1 \cap F, A_2 \cap F, \ldots$, and this contradicts Sierpiński's theorem on continua (see p. 113 of [1] or p. 16 of [5]).

So in trying to construct our example, we look for an example of a locally compact connected space which can be decomposed into a sequence of disjoint closed sets. Such a space was given by Kuratowski on p. 115 of [1]. As it is the essential feature in the construction of our example, we begin Section 3 by describing this space of Kuratowski.

3. In the Euclidean plane let A_n consist of the points (x, y) which satisfy one of the following conditions:

(i) $-1/2^n \leqslant x \leqslant 1,\ y = 1/2^n$,

(ii) $x = -1/2^n,\ -1/2^n \leqslant y \leqslant 1/2^n$,

(iii) $-1/2^n \leqslant x \leqslant 1/2^n,\ y = -1/2^n$,

(iv) $x = 1/2^n,\ -1/2^n \leqslant y \leqslant 0$,

(v) $3/2^{n+2} < x < 3/2^{n+1},\ x \leqslant 1, y = 0$.

Then the set $A = \bigcup_{n=0}^{\infty} A_n$ is the space given by Kuratowski in [1]. It is shown in fig. 1, where the crosses indicate the points on the line segment $0 \leqslant x \leqslant 1, y = 0$ which are not in A.

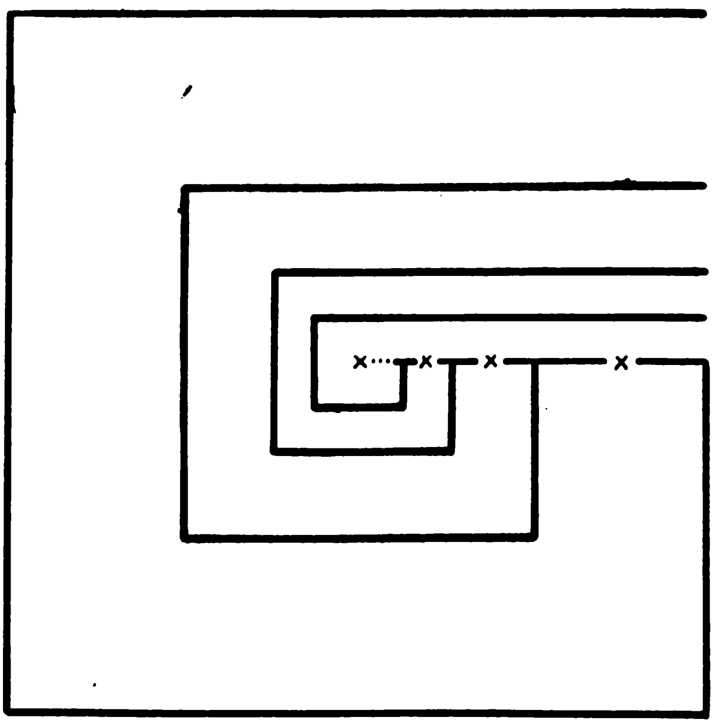

Fig. 1

In order to describe the space of our counter example, we identify the point (x, y) in the Euclidean plane with the point $(x, y, 0)$ in Euclidean 3-space, of which the set A therefore becomes a subset.

Let B_n be the component of $[-1, 1] \times [-1, 1] \times \{0\} - \bar{A}$ whose frontier lies in $\bar{A}_{n-1} \cup \bar{A}_n$ for $n = 1, 2, 3, \ldots$ We define a set Y by subtracting from the cube $[-1, 1]\times[-1, 1]\times[0, -1]$ the two sets $\bigcup_{n=1}^{\infty} B_n \times [0, -1/2^n)$ and $[0, 1]\times\{0\}\times\{0\}-A$. The set Y is shown in fig. 2.

Let Z be the reflection of Y in the plane $z = 0$, and let $X = Y \cup Z$. The space X is our counter example.

4. It is clear that X is a Peano space in which $A_0, A_1, \ldots$ is a sequence of disjoint closed sets such that $X - A_n$ is connected for each n and yet $X - \bigcup_{n=0}^{\infty} A_n$ is not connected. Thus it remains only to show that X is unicoherent.

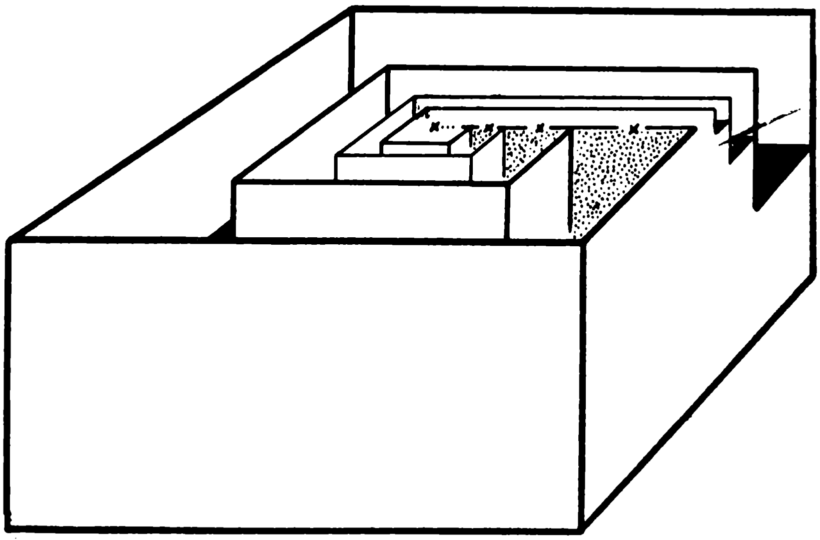

Fig. 2

In order to do this we shall quote three theorems which can be found with small changes in chap. XI of [5]. We first make two definitions.

We denote by S^1 the circle of complex numbers of unit modulus. We say that a space X is *contractible with respect to* S^1 if each mapping $f\colon X \to S^1$ is homotopic to the constant mapping from X into S^1. We say that a space X has *property* (b) if for each mapping $f\colon X \to S^1$ there is a real-valued mapping φ on X such that $f(x) = \exp[i\varphi(x)]$ for each $x \in X$. The first of these definitions may be found in [1]; the second in [5].

We then have the following three theorems, in which it is assumed for convenience that the spaces in question are separable and metric.

THEOREM 1. *A space X is contractible with respect to S^1 if and only if it has property* (b).

THEOREM 2. *Let X_1 and X_2 be closed subsets of their union $X = X_1 \cup X_2$ such that $X_1 \cap X_2$ is connected. Then if X_1 and X_2 both have property* (b), *so does X.*

THEOREM 3. *A connected space X which has property* (b) *is unicoherent.*

Now we show that the space X of Section 3 is unicoherent. We notice that Y has this property: if (x, y, z) belongs to Y so do all the points on the line segment joining (x, y, z) and $(x, y, -1)$. From this it follows that the square $[-1, 1]\times[-1, 1]\times\{-1\}$ is a deformation retract of Y, and so Y is contractible. Therefore Y is contractible with respect to S^1, and so Z is as well. Thus, by theorem 1, both Y and Z have property (b). Since Y and Z are closed subsets of X and $Y \cap Z = A$, it now follows from theorem 2 that X has property (b). Thus, by theorem 3, X is unicoherent.

5. We have seen that the theorem of Section 2 does not hold for an arbitrary Peano space, and yet it does hold for some non-compact Peano spaces, as has been shown by Miss Mullikin and van Est in [3] and [4], respectively. These considerations lead us to seek a precise analytical definition of the class of unicoherent Peano spaces for which the theorem of Section 2 holds.

We notice that the space X of Section 3 has this property: some of its points (namely those of the form $x \neq 3/2^{n+1}$, for $n = 1, 2, 3, \ldots$, $y = 0$, $z = 0$) do not lie in unicoherent regions with compact closures. Since the Euclidean spaces (and likewise the locally Euclidean spaces) do not suffer from this deficiency, we are prompted to ask:

QUESTION. *Let X be a unicoherent Peano space which has a covering by unicoherent regions with compact closures. If $A_1, A_2, \ldots$ is a sequence of disjoint closed sets no one of which separates X, is $X - \bigcup_{n=1}^{\infty} A_n$ necessarily connected?* (**P 744**)

If this fails we can try imposing stronger conditions on the unicoherent regions that cover X. We can for example demand that their closures be unicoherent Peano continua.

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