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A NOTE ON HOMOMORPHISMS OF OPERATOR ALGEBRAS

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Let K be a real Hilbert space (the complex case can be handled in a similar way) and denote by $\boldsymbol{B}(K)$ the Banach algebra of all bounded operators in K. Consider a Banach algebra homomorphism $R: \boldsymbol{B}(K) \to \boldsymbol{B}(H)$ preserving the identities, where H is another Hilbert space. It was proved in [2], Theorem 4.1, that if $\dim(H) < \dim(K)$, then necessarily $\dim(H) = 0$ and $R = 0$, and (loc. cit., Theorem 4.5) that if $\dim(K) = \dim(H) = \aleph_0$, then R is necessarily one-to-one. We intend to show here that this conclusion does not follow when $\dim(K) = \dim(H) > \aleph_0$. More precisely, we have the

THEOREM. *Let K be a Hilbert space of dimension $a \geqslant \aleph_0$. Then there exists a Hilbert space H and a Banach algebra homomorphism $R: \boldsymbol{B}(K) \to \boldsymbol{B}(H)$ such that*

(a) $\dim(H) \leqslant a^{\aleph_0}$;

(b) $\ker(R) = \mathscr{C}(K) =$ *the ideal of compact operators in K.*

We will need two lemmas. In the sequel, $\mathrm{Card}(A)$ will denote the cardinal power of the set A.

LEMMA 1. *If Ω is the set of all sequences in a Hilbert space K, then*

$$\mathrm{Card}(\Omega) \leqslant \big(\dim(K)\big)^{\aleph_0} \cdot 2^{\aleph_0}.$$

Proof. By definition, $\Omega = K^N$ ($N =$ set of positive integers), so that $\mathrm{Card}(\Omega) = \mathrm{Card}(K)^{\aleph_0}$. It remains to compute a bound for $\mathrm{Card}(K)$. Now, K is clearly the union of all closed subspaces of K generated by countable subsets of any orthonormal basis. Hence, if $\{e_a\}_{a\in I}$ is an orthonormal basis of K, we have

$$\mathrm{Card}(K) \leqslant \mathrm{Card}\big(P_c(I)\big)\cdot \mathrm{Card}(l^2),$$

where $P_c(I)$ denotes the set of all countable subsets of I and l^2 is the ordinary space of square summable sequences. Using the axiom of choice, one can define an injection of $P_c(I)$ in I^N. Therefore

$$\mathrm{Card}\big(P_c(I)\big) \leqslant \big(\mathrm{Card}(I)\big)^{\aleph_0}.$$

Furthermore,

$$\mathrm{Card}(l^2) \leqslant \mathrm{Card}(R^N) = 2^{\aleph_0},$$

whence

$$\mathrm{Card}(\Omega) = \mathrm{Card}(K)^{\aleph_0} \leqslant \left(\mathrm{Card}(I)^{\aleph_0} \cdot 2^{\aleph_0}\right)^{\aleph_0} = \dim(K)^{\aleph_0} \cdot 2^{\aleph_0}.$$

Let l^∞ be the space of all bounded real sequences.

Lemma 2. *Let K be a Hilbert space, $c_0(K_w)$ the linear space of all sequences in K which converge weakly to 0. If* LIM: $l^\infty \to R$ *is any positive linear functional on l^∞ that coincides with the ordinary limit when the latter exists, set, for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in $c_0(K_w)$,*

$$[x, y] = \mathrm{LIM}\{(x_n, y_n)\},$$

where $(\,,)$ denotes the inner product in K. Then $p(x) = [x, x]^{1/2}$ is a seminorm on $c_0(K_w)$ and the completion H of the normed space obtained from $c_0(K_w)$ by factoring out the elements x with $p(x) = 0$, is a Hilbert space such that

$$\dim(H) \leqslant \left(\dim(K)\right)^{\aleph_0} \cdot 2^{\aleph_0}.$$

Proof. As H is the completion of a quotient of $c_0(K_w)$, its cardinal $\mathrm{Card}(H)$ can not exceed $\mathrm{Card}\left((c_0(K_w))^N\right) = \left(\mathrm{Card}(c_0(K_w))\right)^{\aleph_0}$. Lemma 2 follows then from Lemma 1 above and the fact that $c_0(K_w) \subset \Omega$.

Consider now an operator $T \in B(K)$. T can be made to act on $c_0(K_w)$ as follows: if $x = \{x_1, x_2, \ldots\}$, then $Tx = \{Tx_1, Tx_2, \ldots\}$. Clearly $Tx \in c_0(K_w)$. Moreover,

$$\left(p(Tx)\right)^2 = \mathrm{LIM}(Tx_n, Tx_n) \leqslant \|T\|^2 \mathrm{LIM}(x_n, x_n),$$

or $p(Tx) \leqslant \|T\| p(x)$. This shows that T defines a bounded operator $R(T)$ on (the quotient $c_0(K_w)/\{x; p(x) = 0\}$, and hence on) H, and in fact, that also $\|R(T)\| \leqslant [\![T[\![$. Obviously R: $\boldsymbol{B}(K) \to \boldsymbol{B}(H)$ is a homomorphism of Banach algebras. Since for $C \in \boldsymbol{B}(K)$ compact and $x = \{x_n\} \in c_0(K_w)$ we have $|Cx_n| \to 0$, it follows that $p(Cx) = 0$ for all x and therefore $R(C) = 0$. On the other hand, if $R(T) = 0$ and $x = \{x\ \} \in c_0(K_w)$, let $\varepsilon = \limsup |Tx_n|$. Clearly, there exists a subsequence $x' = \{x_n'\}$ of $\{x_n\}$ such that $\varepsilon = \lim |Tx_n'|$. But then $\mathrm{LIM}|Tx_n'| = \varepsilon$ and also $0 = p(Tx') = \varepsilon$. Therefore $|Tx_n| \to 0$ and T maps weakly convergent sequences into norm convergent sequences, which implies that T is compact. It follows that the kernel of R is precisely the ideal of compact operators in K.

The theorem follows from these two lemmas and the remark that if $a \geqslant \aleph_0$, then $a^{\aleph_0} \cdot 2^{\aleph_0} = a^{\aleph_0}$.

Corollary. *If $\dim(K) = a \geqslant \aleph_0$ and $a^{\aleph_0} = a$, then there are endomorphisms of the Banach algebra $\boldsymbol{B}(K)$ that are not one-to-one.*

Remark. The construction of H in Lemma 2 is due to Calkin (see [1]). Furthermore, all the theorems in § 5 of [1] can be generalized to non-separable Hilbert spaces, from which it follows that $R(T)$ is either 0 or not compact and that the homomorphism R: $\boldsymbol{B}(K)/\mathscr{C}(K) \to \boldsymbol{B}(H)$ induced by R, is an isometry. Of course, $R(T^*) = (R(T))^*$, where * denotes adjoints.

REFERENCES

[1] J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space*, Annals of Mathematics 42 (1941), p. 839-873.

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