COLLOQUIUM MATHEMATICUM

VOL. XLVI 1982 FASC. 2

DIVIDING AN ARC TO SUBARCS WITH EQUAL CHORDS

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The purpose of this note is to prove the following

THEOREM. *Let (A, d) be a metric arc with end points a and b. Then for every positive integer n there is a sequence $a = x_0, x_1, \ldots, x_n = b$ of consecutive points in A such that $d(x_{j-1}, x_j)$ is independent of j.*

Proof. The case $n = 1$ is trivial and the case $n = 2$ is easy. The case $n \geqslant 3$ is somewhat harder than what might be expected. We may assume that A is the interval $[0, 1]$ and that $d(0, 1) = 1$, but the metric d is not the standard one. Let $e_1, \ldots, e_n$ be the standard orthonormal basis of R^n and let $E \subset R^{n-1}$ be the $(n-1)$-simplex with vertices $v_j = e_j + \ldots + e_{n-1}$ and $v_n = 0$. Thus

$$E = \{x \in R^{n-1} \mid 0 \leqslant x_1 \leqslant \ldots \leqslant x_{n-1} \leqslant 1\}.$$

Define $f\colon E \to R^n$ by $f_j(x) = d(x_{j-1}, x_j)$, where we put $x_0 = 0$ and $x_n = 1$. Suppose that the Theorem is false. Then fE does not meet the line $L = \{y \in R^n \mid y_1 = \ldots = y_n\}$.

Set $M = \{y \in R^n \mid y_j \geqslant 0 \text{ for all } j,\ y \neq 0\}$ and $\partial M = M \setminus \operatorname{int} M$. Then f defines a map $f_1\colon (E, \partial E) \to (M, \partial M)$. More precisely, $f(v_j) = e_j$, and the $(n-2)$-face of E opposite to v_j is mapped into the plane $y_j = 0$. Using a projection from L, we see that f_1 is homotopic to a map $g\colon (E, \partial E) \to (M, \partial M)$ with $gE \subset \partial M$. Thus f_1 is homologically trivial. Let $\Delta \subset R^n$ be the $(n-1)$-simplex with vertices $e_1, \ldots, e_n$. Then $r(y) = y/(y_1 + \ldots + y_n)$ defines a retraction $r\colon (M, \partial M) \to (\Delta, \partial\Delta)$. The restriction $u\colon \partial E \to \partial\Delta$ of rf_1 is homotopic to the affine homeomorphism $h\colon \partial E \to \partial\Delta$ with $h(v_j) = e_j$ by the segmental homotopy $(x, t) \mapsto (1-t)u(x) + th(x)$. For the homology groups with integral coefficients we have the commutative diagram

$$\begin{tikzcd} H_{n-1}(E, \partial E) \arrow[r, "\partial"] \arrow[d, "(rf_1)_*"] & H_{n-2}(\partial E) \arrow[d, "u_*"] \\ H_{n-1}(\Delta, \partial\Delta) \arrow[r, "\partial"] & H_{n-2}(\partial\Delta) \end{tikzcd}$$

where $(rf_1)_*$ is the zero map and all other maps are isomorphisms between infinite cyclic groups. This contradiction proves the Theorem.

Remarks. (1) Given any sequence $t_1, \ldots, t_{n-1}$ of positive numbers, one can similarly choose the points x_j so that

$$d(x_j, x_{j+1})/d(x_{j-1}, x_j) = t_j \quad \text{for all } j.$$

In the proof, it is only necessary to replace the line L by the line $\{y \mid y_{j+1} = t_j y_j \text{ for all } j\}$.

(2) One can replace the distance $d(x_{j-1}, x_j)$ in the Theorem by the diameter of the subarc with end points x_{j-1}, x_j or, more generally, by any non-negative continuous function $f(x_{j-1}, x_j)$ which vanishes for $x_{j-1} = x_j$.

(3) As $n \to \infty$ in the Theorem, the distances $d(x_{j-1}, x_j)$ converge to zero. To show this, let $\varepsilon > 0$. Assume again that $A = [0, 1]$. Since d is uniformly continuous, there is a $\delta > 0$ such that $d(x, y) < \varepsilon$ whenever $x, y \in A$ and $|x-y| < \delta$. If $n > 1/\delta$, then there are points x_{j-1}, x_j with $|x_j - x_{j-1}| < \delta$. Then $d(x_{j-1}, x_j) < \varepsilon$.

COROLLARY. *Let (X, d) be an arcwise connected metric space and let $a, b \in X$. Then for every positive integer n there is a sequence of distinct points $a = x_0, \ldots, x_n = b$ in X such that $d(x_{j-1}, x_j)$ is independent of j.*

QUESTION. Is the Corollary true if arcwise connectedness is replaced by connectedness? (**P 1260**)

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Reçu par la Rédaction le 20. 4. 1979