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A REMARK ON THE ORBIT SPACES UNDER MULTIPLICATIVE GROUP ACTIONS

BY

JERZY JURKIEWICZ (WARSZAWA)

The present paper is suggested by the following question:

Let the ground field k be algebraically closed. Consider the k^*-action on $\boldsymbol{P}_3$ and assume that

(i) the fixed point set is finite.

Can any Hirzebruch surface F_m be the orbit space of an invariant open subset of $\boldsymbol{P}_3$?

The answer is negative. We prove a more general

THEOREM. *Let k^* act on a projective space $\boldsymbol{P}_n$, where $n \geqslant 3$, and assume* (i) *holds. Let $\mathcal{U} \subset \boldsymbol{P}_n$ be an invariant open set. If the orbit space $\mathcal{U}/k^*$ exists and is complete, then it is singular.*

In the example below we treat the case without restriction (i).

The proof of the theorem is based on two theories: torus embeddings (see [2]) and sectional sets [1]. By a *cone* we will mean a convex rational polyhedral cone, and by a *conical complex* a "finite rational partial decomposition" (see [2], Sections 1 and 2). By a *d-cone* we will understand a d-dimensional cone. Let $T \subset X_\Sigma$ be the embedding of the n-dimensional torus T, corresponding to a conical complex Σ. Assume X_Σ is complete and smooth. Let N be the lattice of the 1-parameter subgroups (1-P.S.) of T, let a be an element of N which is primitive, i.e. not a multiple of an element of N, and let λ_a be the corresponding 1-P.S.: $k^* \to T$. Define a k^*-action on X_Σ by

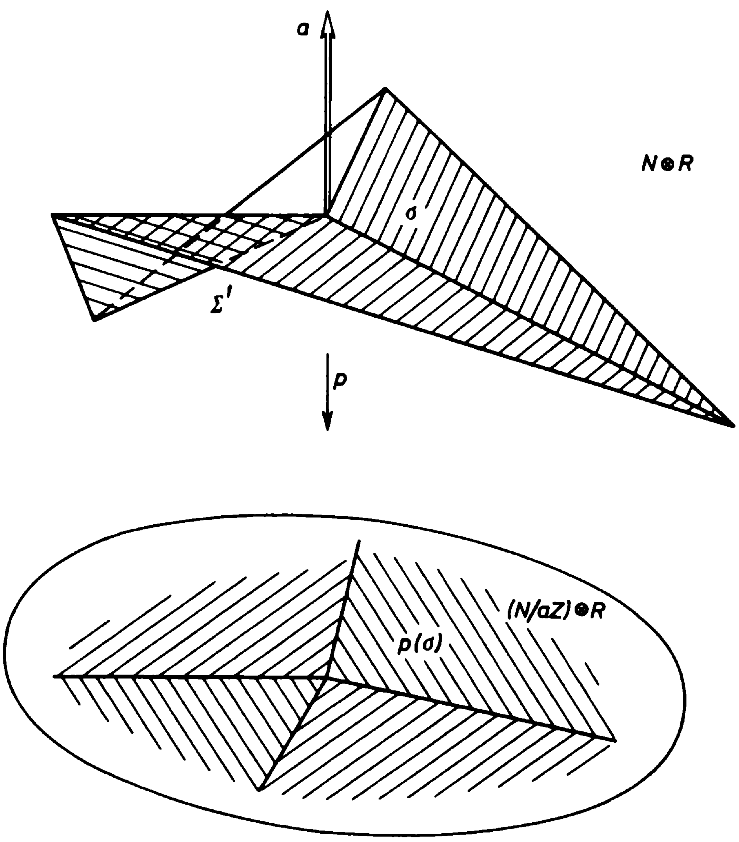
(ii) $tx = \lambda_a(t)x$ (the canonical action of T on X_Σ is used).

Assume the k^* fixed point set is finite (then it coincides with the T fixed point set), say $\{x_1, x_2, \ldots, x_m\}$. The equivalent assumption is the following: a does not belong to the linear hull of any $(n-1)$-cone of Σ. Consider a cross-section $\{x_1, \ldots, x_i\}$, $\{x_{i+1}, \ldots, x_m\}$ of the fixed point set (see [1]). This means by definition that if a k^*-orbit ends at x_j, where $j \leqslant i$, then it begins at x_l, where $l \leqslant i$. For $j = 1, 2, \ldots, m$ let $\sigma_j \in \Sigma$ be the n-cone corresponding to the fixed point x_j. Denote by $\sigma_1, \ldots, \sigma_i$ lower n-cones, and by $\sigma_{i+1}, \ldots, \sigma_m$ upper n-cones. Let $\mathcal{U}$ be the open sectional set corresponding to the given cross-section [1]. Recall that $\mathscr{U}$ is the union of orbits beginning below and ending above the cross-section.

A standard argument gives

PROPOSITION 1. *$\mathscr{U}$ is a torus embedding of the form $X_{\Sigma'}$, where Σ' is a closed subcomplex of Σ consisting of the common faces of the lower n-cones and upper n-cones.*

Observe that the maximal dimension of the cones of Σ' is $n-1$. The element a does not belong to any cone of Σ'.



PROPOSITION 2. *Let σ be an l-cone ($l < n$), X_σ the corresponding affine embedding, $a \in N \setminus \operatorname{lin}(\sigma)$ a primitive element, and consider the k^*-action on X_σ defined by* (ii). *Then the orbit space X_σ/k^* exists; it is an affine embedding of the $(n-1)$-torus T/T_0, where $T_0 = \lambda_a(k^*)$, and the corresponding cone is the image of σ under the projection p: $N \otimes R \to (N/aZ) \otimes R$.*
(Note that the mapping p: $\sigma \to p(\sigma)$ is a bijection.)

Sketch of the proof. Since there are no fixed points in X_σ, the categorical quotient is the orbit space. Let A_σ be the affine algebra of X_σ. Then the affine algebra of X_σ/k^* is $A_\sigma^{k^*}$, the invariant algebra. The remaining argument is standard.

Proof of the Theorem. By the main theorem of [1], if the orbit space $\mathscr{U}/k^*$ exists and is complete, then $\mathscr{U}$ is a sectional set. There are $n+1$

$\geqslant 4$ fixed points in P_n. Consider the cross-section defining $\mathscr{U}$. Replacing if necessary the given k^*-action by the opposite one we may assume that the upper part contains at least 2 fixed points. Thus the cross-section has the form $\{x_0, \ldots\}$, $\{\ldots, x_{n-1}, x_n\}$, where x_i are the fixed points, and in P_n there exists an orbit beginning at x_i and ending at x_j if and only if $i < j$. The point x_0 is the source and x_n is the sink.

Let $(z_0 : \ldots : z_n)$ be the coordinate system in P_n such that

$$x_i = (0:0:\ldots:\underset{i}{1}:0).$$

It follows from the previous choices that the k^*-action can be written in the form

$$t(z_0 : z_1 : \ldots : z_n) = (t^{a_0} z_0 : t^{a_1} z_1 : \ldots : t^{a_n} z_n),$$

where

(iii) $0 = a_0 < a_1 < \ldots < a_n$.

Define

$$T = \{z \in P_n : z_i \neq 0 \text{ for all } i\}$$

with the ordinary multiplication. Then P_n is an embedding of the torus T and the corresponding complex Σ is spanned by elements $e_0, e_1, \ldots, e_n$ of N such that $\sum e_i = 0$ and $\{e_1, \ldots, e_n\}$ is a basis of N. The fixed point x_i corresponds to the n-cone σ_i spanned by all e_j, $j \neq i$. The k^*-action originates from the element $a = a_1 e_1 + \ldots + a_n e_n$. The subcomplex Σ' corresponding to $\mathscr{U}$ contains, by Proposition 1, the cones $\tau = \sigma_0 \cap \sigma_{n-1}$ and $\omega = \sigma_0 \cap \sigma_n$. Apply to those two cones Proposition 2 and [2], Theorem 4 in Section 1. The result is that if the both orbit spaces X_τ/k^* and X_ω/k^* were smooth, then the both sets $\{e_1, \ldots, e_{n-2}, a, e_n\}$ and $\{e_1, \ldots, e_{n-1}, a\}$ would be bases of N. Since

$$a = \sum_{i=1}^{n} a_i e_i,$$

this means that $a_{n-1} = \pm 1$ and $a_n = \pm 1$, which contradicts (iii). Hence $\mathscr{U}/k^* = X_{\Sigma'}/k^*$ is singular.

Now we allow nonisolated fixed points. Obviously, we can obtain the projective $(n-1)$-space as an orbit space of an open invariant subset of P_n. In the case $n = 3$ the orbit spaces $\mathscr{U}/k^*$ are torus embeddings of dimension 2, which admit a 4-element affine covering. On the other hand, the only such complete, smooth embeddings, apart from the projective plane, are the surfaces F_m, $m = 0, 1, \ldots$ In fact, each F_m is of the form $\mathscr{U}/k^*$, where $\mathscr{U}$ is an open invariant set in P_3.

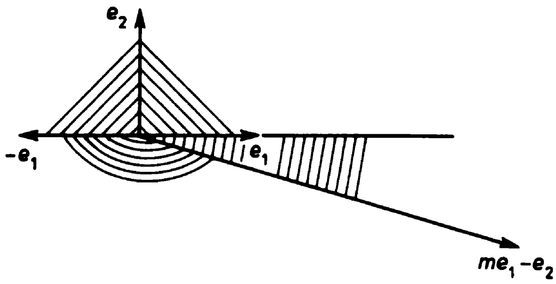
EXAMPLE. Fix m. In the previous notation let $a = -me_1 + e_2 + e_3$. Let $\Sigma' \subset \Sigma$ be the closed subcomplex consisting of 2-cones

$$\langle\langle e_1, e_2\rangle\rangle,\ \langle\langle e_1, e_3\rangle\rangle,\ \langle\langle e_0, e_2\rangle\rangle,\ \langle\langle e_0, e_3\rangle\rangle$$

and their faces. Take the new basis $\{e_1, e_2, a\}$ of N and the basis $\{e_1, e_2\}$ of N/aZ. Under the projection $p\colon N \to N/aZ$, we have

$$p(e_0) = -(1+m)e_1, \qquad p(e_3) = me_1 - e_2.$$

Thus, by Proposition 2, the orbit space $X_{\Sigma'}/k^*$ is the torus embedding corresponding to the complex



in N/aZ, and hence $X_{\Sigma'}/k^* \simeq F_m$. For $m = 0$ we obtain $\boldsymbol{P}_1 \times \boldsymbol{P}_1$.

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
WARSZAWA

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